Eriksson's numbers game on certain edge-weighted three-node cyclic graphs

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Abstract

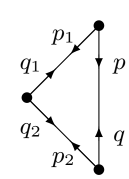
The numbers game is a one-player game played on a finite simple graph with certain "amplitudes" assigned to its edges and with an initial assignment of real numbers to its nodes. The moves of the game successively transform the numbers at the nodes using the amplitudes in a certain way. This game and its interactions with Coxeter/Weyl group theory and Lie theory have been studied by many authors. Following Eriksson, we allow the amplitudes on graph edges to be certain real numbers. Games played on such graphs are "E-games." We show that for certain such three-node cyclic graphs, any numbers game will diverge when played from an initial assignment of nonnegative real numbers not all zero. This result is a key step in a Dynkin diagram classification (obtained elsewhere) of all E-game graphs which meet a certain finiteness requirement.

Keywords: numbers game, Coxeter/Weyl group, Dynkin diagram

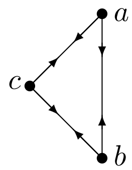
The numbers game is a one-player game played on a finite simple graph with weights (which we call "amplitudes") on its edges and with an initial assignment of real numbers to its nodes. Each of the two edge amplitudes (one for each direction) will be certain negative real numbers. The move a player can make is to "fire" one of the nodes with a positive number. This move transforms the number at the fired node by changing its sign, and it also transforms the number at each adjacent node in a certain way using an amplitude along the incident edge. The player fires the nodes in some sequence of the player's choosing, continuing until no node has a positive number. This game was formulated by Mozes [Moz] for graphs with integer amplitudes. It has also been studied by Proctor [Pro1], [Pro2], Björner [Björ], and Wildberger [Wil1], [Wil2], [Wil3]. The game is the subject of §4.3 of the book [BB] by Björner and Brenti. The numbers game facilitates computations with Coxeter groups and their geometric representations (e.g. see §4.3 of [BB]). See [Don] for discussion of further connections and applications, where mainly we draw on Eriksson's ground-breaking work in [Erik1], [Erik2], and [Erik3].

Our purpose here is to show in Proposition 1 that certain three-node cyclic "E-GCM graphs" (see [Don] for a definition) are not "admissible": that is, any numbers game played on such a graph from a nontrivial initial assignment of nonnegative numbers will not terminate. Our main interest in this proposition is that it furnishes a key step for the proof given in [Don] of the following Dynkin diagram classification result: A connected E-GCM graph has a nontrivial initial assignment of nonnegative numbers such that the numbers game terminates in a finite number of steps if and only if it is a connected "E-Coxeter graph" corresponding to an irreducible finite Coxeter group. (Another proof of this classification result is given in [DE].) All further motivation, definitions, and preliminary results needed to understand the statement and proof of Proposition 1 are given in [Don].

Proposition 1 *Suppose (Γ, M) is the following three-node E-GCM graph:*



Assume that all node pairs are odd-neighborly. Then (Γ, M) is not admissible.

Proof. The proof is somewhat tedious. With amplitudes as depicted in the proposition statement, assign numbers a, b, and c as follows:  Call this position $\lambda = (a, b, c)$, so a is the number at node γ_1, b is at node γ_2, and c is at node γ_3. Without loss of generality, assume that $pq \leq p_1 q_1$ and that $pq \leq p_2 q_2$. Set

$$\kappa_1 := \frac{pp_2 + p_1\sqrt{pq}}{\sqrt{pq}(2-\sqrt{pq})} \qquad \text{and} \qquad \kappa_2 := \frac{qp_1 + p_2\sqrt{pq}}{\sqrt{pq}(2-\sqrt{pq})}.$$

Assume that $a \geq 0$, $b \geq 0$, $c \leq 0$, and that $\left(\kappa_1 - \dfrac{p}{q_2\sqrt{pq}}\right) a + \left(\kappa_2 - \dfrac{q}{q_1\sqrt{pq}}\right) b + c > 0$. These hypotheses will be referred to as condition $(*)$. Notice that a and b cannot both be zero under condition $(*)$. A justification of the following claim will be given at the end of the proof:

Claim: *Under condition $(*)$ there is a sequence of legal node firings from initial position $\lambda = (a, b, c)$ which results in the position $\lambda' = (a', b', c') = (\frac{-q}{\sqrt{pq}}b, \frac{-p}{\sqrt{pq}}a, \kappa_1 a + \kappa_2 b + c)$.*

In this case, observe that $a' \leq 0$, $b' \leq 0$, and $c' > 0$. Now fire at node γ_3 to obtain the position $\lambda^{(1)} = (a_1, b_1, c_1)$ with $a_1 = q_1[\kappa_1 a + (\kappa_2 - \frac{q}{q_1\sqrt{pq}})b + c]$, $b_1 = q_2[(\kappa_1 - \frac{p}{q_2\sqrt{pq}})a + \kappa_2 b + c]$, and $c_1 = -(\kappa_1 a + \kappa_2 b + c)$. Now condition $(*)$ implies that $a_1 > 0$, $b_1 > 0$, and $c_1 < 0$. At this point to see that $\lambda^{(1)} = (a_1, b_1, c_1)$ itself meets condition $(*)$, we only need to show that $\left(\kappa_1 - \dfrac{p}{q_2\sqrt{pq}}\right) a_1 + \left(\kappa_2 - \dfrac{q}{q_1\sqrt{pq}}\right) b_1 + c_1 > 0$. As a first step, we argue that (i) $q_1(\kappa_1 - \frac{p}{q_2\sqrt{pq}}) \geq 1$ and that (ii) $q_2(\kappa_2 - \frac{q}{q_1\sqrt{pq}}) \geq 1$. We only show (i) since (ii) follows by similar reasoning. (From the inequalities (i) and (ii), a third inequality (iii) follows immediately: $q_1(\kappa_1 - \frac{p}{q_2\sqrt{pq}}) + q_2(\kappa_2 - \frac{q}{q_1\sqrt{pq}}) - 1 > 0$.) For the first of the inequalities (i), note that since $1 \leq pq$, then $2 - \sqrt{pq} \leq pq$. Since $pq \leq p_2 q_2$, then $2 - \sqrt{pq} \leq p_2 q_2$. (Similarly $2 - \sqrt{pq} \leq p_1 q_1$.) Thus $\frac{p_2 q_2}{2-\sqrt{pq}} - 1 \geq 0$, and hence $\frac{p_2}{2-\sqrt{pq}} - \frac{1}{q_2} \geq 0$. Therefore, $\frac{q_1 p p_2}{\sqrt{pq}(2-\sqrt{pq})} - \frac{q_1 p}{q_2\sqrt{pq}} \geq 0$. Since $\frac{p_1 q_1 \sqrt{pq}}{\sqrt{pq}(2-\sqrt{pq})} \geq 1$, then $\frac{q_1 p p_2}{\sqrt{pq}(2-\sqrt{pq})} + \frac{p_1 q_1\sqrt{pq}}{\sqrt{pq}(2-\sqrt{pq})} - \frac{q_1 p}{q_2\sqrt{pq}} \geq 1$. From this we get $q_1(\kappa_1 - \frac{p}{q_2\sqrt{pq}}) \geq 1$, which is (i). The following identity is easy to verify:

$$\begin{aligned}
&\left(\kappa_1 - \frac{p}{q_2\sqrt{pq}}\right) a_1 + \left(\kappa_2 - \frac{q}{q_1\sqrt{pq}}\right) b_1 + c_1 \\
&\qquad = \left(\kappa_1 - \frac{p}{q_2\sqrt{pq}}\right)\left[q_1\left(\kappa_1 - \frac{p}{q_2\sqrt{pq}}\right) + q_2\left(\kappa_2 - \frac{q}{q_1\sqrt{pq}}\right) - 1\right] a_1 \\
&\qquad\quad + \left(\kappa_2 - \frac{q}{q_1\sqrt{pq}}\right)\left[q_1\left(\kappa_1 - \frac{p}{q_2\sqrt{pq}}\right) + q_2\left(\kappa_2 - \frac{q}{q_1\sqrt{pq}}\right) - 1\right] b_1 \\
&\qquad\quad + \left[q_1\left(\kappa_1 - \frac{p}{q_2\sqrt{pq}}\right) + q_2\left(\kappa_2 - \frac{q}{q_1\sqrt{pq}}\right) - 1\right] c_1 \\
&\qquad\quad + \frac{p}{q_2\sqrt{pq}}\left[q_1\left(\kappa_1 - \frac{p}{q_2\sqrt{pq}}\right) - 1\right] a_1 + \frac{q}{q_1\sqrt{pq}}\left[q_2\left(\kappa_2 - \frac{q}{q_1\sqrt{pq}}\right) - 1\right] b_1
\end{aligned}$$

Now the inequalities (i), (ii), and (iii) of the previous paragraph together with the inequality $\left(\kappa_1 - \dfrac{p}{q_2\sqrt{pq}}\right) a + \left(\kappa_2 - \dfrac{q}{q_1\sqrt{pq}}\right) b + c > 0$ from condition $(*)$ imply that $\left(\kappa_1 - \dfrac{p}{q_2\sqrt{pq}}\right) a_1 +$

$\left(\kappa_2 - \dfrac{q}{q_1\sqrt{pq}}\right) b_1 + c_1 > 0$, as desired. This means that position $\lambda^{(1)} = (a_1, b_1, c_1)$ meets condition $(*)$ and none of its numbers are zero. In view of our **Claim**, we may apply to position $\lambda^{(1)}$ a legal sequence of node firings followed by firing node γ_3 as before to obtain a position $\lambda^{(2)} = (a_2, b_2, c_2)$ that meets condition $(*)$ with none of its numbers zero, etc. So from any such $\lambda = (a, b, c)$ we have a divergent game sequence. In view of inequalities (i) and (ii), the fundamental positions $\omega_1 = (1, 0, 0)$ and $\omega_2 = (0, 1, 0)$ meet condition $(*)$. The fundamental position $\omega_3 = (0, 0, 1)$ does not meet condition $(*)$. However, by firing at node γ_3 we obtain the position $(q_1, q_2, -1)$, which meets condition $(*)$ by inequality (iii). Thus from any fundamental position there is a divergent game sequence, and so by Lemma 2.5 of [Don] the three-node E-GCM graph we started with is not admissible.

To complete the proof we must justify our **Claim**. Beginning with position $\lambda = (a, b, c)$ under condition $(*)$, we propose to fire at nodes γ_1 and γ_2 in alternating order until this is no longer possible. We assert that the resulting position will be $\lambda' = (a', b', c') = (\frac{-q}{\sqrt{pq}}b, \frac{-p}{\sqrt{pq}}a, \kappa_1 a + \kappa_2 b + c)$. There are three cases to consider: (I), a and b are both positive, (II), $a > 0$ and $b = 0$, and (III), $a = 0$ and $b > 0$. For (I), we wish to show that $(\gamma_1, \gamma_2, \ldots, \gamma_1)$ of length m_{12} is a sequence of legal node firings. That is, we must check that

$$\begin{array}{llcll} (1) & \langle (s_2 s_1)^k.\lambda, \alpha_1 \rangle & = & \langle \lambda, (s_1 s_2)^k.\alpha_1 \rangle > 0 & \text{for } 0 \leq k \leq (m_{12} - 1)/2, \text{ and} \\ (2) & \langle s_1 (s_2 s_1)^k.\lambda, \alpha_2 \rangle & = & \langle \lambda, s_1 (s_2 s_1)^k.\alpha_2 \rangle > 0 & \text{for } 0 \leq k < (m_{12} - 1)/2 \end{array}$$

For (II), we wish to show that $(\gamma_1, \gamma_2, \ldots, \gamma_1, \gamma_2)$ of length $m_{12} - 1$ is a sequence of legal node firings. That is, we must check that

$$\begin{array}{llcll} (3) & \langle (s_2 s_1)^k.\lambda, \alpha_1 \rangle & = & \langle \lambda, (s_1 s_2)^k.\alpha_1 \rangle > 0 & \text{for } 0 \leq k < (m_{12} - 1)/2, \text{ and} \\ (4) & \langle s_1 (s_2 s_1)^k.\lambda, \alpha_2 \rangle & = & \langle \lambda, s_1 (s_2 s_1)^k.\alpha_2 \rangle > 0 & \text{for } 0 \leq k < (m_{12} - 1)/2 \end{array}$$

For (III), we wish to show that $(\gamma_2, \gamma_1, \ldots, \gamma_2, \gamma_1)$ of length $m_{12} - 1$ is a sequence of legal node firings. That is, we must check that

$$\begin{array}{llcll} (5) & \langle (s_1 s_2)^k.\lambda, \alpha_2 \rangle & = & \langle \lambda, (s_2 s_1)^k.\alpha_2 \rangle > 0 & \text{for } 0 \leq k < (m_{12} - 1)/2, \text{ and} \\ (6) & \langle s_2 (s_1 s_2)^k.\lambda, \alpha_1 \rangle & = & \langle \lambda, s_2 (s_1 s_2)^k.\alpha_1 \rangle > 0 & \text{for } 0 \leq k < (m_{12} - 1)/2 \end{array}$$

To address (1) through (6), we consider matrix representations for each $S_i := \sigma_M(s_i)$ (where $i = 1, 2, 3$) under the representation σ_M. With respect to the ordered basis $\mathfrak{B} = (\alpha_1, \alpha_2, \alpha_3)$ for V we have $X_1 := [S_1]_{\mathfrak{B}} = \begin{pmatrix} -1 & p & p_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $X_2 := [S_2]_{\mathfrak{B}} = \begin{pmatrix} 1 & 0 & 0 \\ q & -1 & p_2 \\ 0 & 0 & 1 \end{pmatrix}$, and so

$$X_{1,2} := [S_1 S_2]_{\mathfrak{B}} = X_1 X_2 = \begin{pmatrix} pq - 1 & -p & p_2 p + p_1 \\ q & -1 & p_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$X_{2,1} := [S_2 S_1]_{\mathfrak{B}} = X_2 X_1 = \begin{pmatrix} -1 & p & p_1 \\ -q & pq - 1 & p_1 q + p_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

For (1) through (6) above, we need to understand $X_{1,2}^k\begin{pmatrix}1\\0\\0\end{pmatrix}$, $X_2X_{1,2}^k\begin{pmatrix}1\\0\\0\end{pmatrix}$, $X_{2,1}^k\begin{pmatrix}0\\1\\0\end{pmatrix}$, and $X_1X_{2,1}^k\begin{pmatrix}0\\1\\0\end{pmatrix}$. Set $\theta := \pi/m_{12}$. Then we can write $X_{1,2} = PDP^{-1}$ for nonsingular P and diagonal matrix D as in

$$\frac{1}{q(e^{2i\theta}-e^{-2i\theta})}\begin{pmatrix} e^{2i\theta}+1 & e^{-2i\theta}+1 & p_2p+2p_1 \\ q & q & p_1q+2p_2 \\ 0 & 0 & 4-pq \end{pmatrix}\begin{pmatrix} e^{2i\theta} & 0 & 0 \\ 0 & e^{-2i\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} q & -e^{-2i\theta}-1 & C_1 \\ -q & e^{2i\theta}+1 & C_2 \\ 0 & 0 & C_3 \end{pmatrix}$$

where $C_1 = [-q(p_2p+2p_1)+(e^{-2i\theta}+1)(p_1q+2p_2)]/(4-pq)$, $C_2 = [q(p_2p+2p_1)-(e^{2i\theta}+1)(p_1q+2p_2)]/(4-pq)$, and $C_3 = q(e^{2i\theta}-e^{-2i\theta})/(4-pq)$. With some work we can calculate $X_{1,2}^k$, which results in

$$X_{1,2}^k = \begin{pmatrix} \frac{\sin(2(k+1)\theta)+\sin(2k\theta)}{\sin(2\theta)} & \frac{-p\sin(2k\theta)}{\sin(2\theta)} & C_1' \\ \frac{q\sin(2k\theta)}{\sin(2\theta)} & \frac{-\sin(2k\theta)-\sin(2(k-1)\theta)}{\sin(2\theta)} & C_2' \\ 0 & 0 & 1 \end{pmatrix},$$

with

$$C_1' = -\frac{p_2p+2p_1}{4-pq}\left[\frac{\sin(2(k+1)\theta)+\sin(2k\theta)}{\sin(2\theta)}-1\right]+\frac{p(p_1q+2p_2)\sin(2k\theta)}{(4-pq)\sin(2\theta)}$$

and

$$C_2' = -\frac{q(p_2p+2p_1)\sin(2k\theta)}{(4-pq)\sin(2\theta)}+\frac{p_1q+2p_2}{4-pq}\left[\frac{\sin(2k\theta)+\sin(2(k-1)\theta)}{\sin(2\theta)}+1\right].$$

Similar reasoning (or simply interchanging the roles of α_1 and α_2 in the preceding calculations, or noting that $X_{2,1}^k = (X_{1,2}^{-1})^k = X_{1,2}^{-k}$) shows that

$$X_{2,1}^k = \begin{pmatrix} \frac{-\sin(2k\theta)-\sin(2(k-1)\theta)}{\sin(2\theta)} & \frac{p\sin(2k\theta)}{\sin(2\theta)} & C_1'' \\ \frac{-q\sin(2k\theta)}{\sin(2\theta)} & \frac{\sin(2(k+1)\theta)+\sin(2k\theta)}{\sin(2\theta)} & C_2'' \\ 0 & 0 & 1 \end{pmatrix},$$

with

$$C_1'' = \frac{p_2p+2p_1}{4-pq}\left[\frac{\sin(2k\theta)+\sin(2(k-1)\theta)}{\sin(2\theta)}+1\right]-\frac{p(p_1q+2p_2)\sin(2k\theta)}{(4-pq)\sin(2\theta)}$$

and

$$C_2'' = \frac{q(p_2p+2p_1)\sin(2k\theta)}{(4-pq)\sin(2\theta)}-\frac{p_1q+2p_2}{4-pq}\left[\frac{\sin(2(k+1)\theta)+\sin(2k\theta)}{\sin(2\theta)}-1\right].$$

Then

$$X_2X_{1,2}^k = \begin{pmatrix} \frac{\sin(2(k+1)\theta)+\sin(2k\theta)}{\sin(2\theta)} & \frac{-p\sin(2k\theta)}{\sin(2\theta)} & C_1' \\ \frac{q\sin(2(k+1)\theta)}{\sin(2\theta)} & \frac{(1-pq)\sin(2k\theta)+\sin(2(k-1)\theta)}{\sin(2\theta)} & qC_1'-C_2'+p_2 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$X_1X_{2,1}^k = \begin{pmatrix} \frac{(1-pq)\sin(2k\theta)+\sin(2(k-1)\theta)}{\sin(2\theta)} & \frac{p\sin(2(k+1)\theta)}{\sin(2\theta)} & -C_1''+pC_2''+p_1 \\ \frac{-q\sin(2k\theta)}{\sin(2\theta)} & \frac{\sin(2(k+1)\theta)+\sin(2k\theta)}{\sin(2\theta)} & C_2'' \\ 0 & 0 & 1 \end{pmatrix}.$$

Now we can justify (1) through (6). For example, for (4) we see that since $X_1X_{2,1}^k\begin{pmatrix}0\\1\\0\end{pmatrix}$ is the second column of the matrix $X_1X_{2,1}^k$, then $\langle\lambda, s_1(s_2s_1)^k.\alpha_2\rangle = a\frac{p\sin(2(k+1)\theta)}{\sin(2\theta)}$, which is positive since $a > 0$, $p > 0$, and (recalling that m_{12} is odd) $2(k+1) < m_{12}$.

Then the proposed firing sequence for each of cases (I), (II), and (III) is legal. To see in case (I) that the resulting position is the claimed $\lambda' = (a', b', c') = (\frac{-q}{\sqrt{pq}}b, \frac{-p}{\sqrt{pq}}a, \kappa_1 a + \kappa_2 b + c)$, we need to calculate $\langle s_1(s_2s_1)^k.\lambda, \alpha_i\rangle = \langle\lambda, s_1(s_2s_1)^k.\alpha_i\rangle$ for each of $i = 1, 2, 3$, where k is now $(m_{12}-1)/2$. With patience one can confirm that

$$X_1X_{2,1}^k = \begin{pmatrix} 0 & -p/\sqrt{pq} & \kappa_1 \\ -q/\sqrt{pq} & 0 & \kappa_2 \\ 0 & 0 & 1 \end{pmatrix},$$

from which the claim follows. Similar computations confirm the claim for cases (II) and (III). $\square$

References

[Björ] A. Björner, "On a combinatorial game of S. Mozes," preprint, 1988.

[BB] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Springer, New York, 2005.

[Don] R. G. Donnelly, "Eriksson's numbers game and finite Coxeter groups," *European J. Combin.*, to appear. `arXiv:0710.0173`.

[DE] R. G. Donnelly and K. Eriksson, "The numbers game and Dynkin diagram classification results," preprint.

[Erik1] K. Eriksson, "Strongly Convergent Games and Coxeter Groups," Ph.D. thesis, KTH, Stockholm, 1993.

[Erik2] K. Eriksson, "The numbers game and Coxeter groups," *Discrete Math.* **139** (1995), 155–166.

[Erik3] K. Eriksson, "Strong convergence and a game of numbers," *European J. Combin.* **17** (1996), 379–390.

[Moz] S. Mozes, "Reflection processes on graphs and Weyl groups," *J. Combin. Theory Ser. A* **53** (1990), 128–142.

[Pro1] R. A. Proctor, "Bruhat lattices, plane partition generating functions, and minuscule representations," *European J. Combin.* **5** (1984), 331-350.

[Pro2] R. A. Proctor, "Minuscule elements of Weyl groups, the numbers game, and d-complete posets," *J. Algebra* **213** (1999), 272-303.

[Wil1] N. J. Wildberger, "A combinatorial construction for simply-laced Lie algebras," *Adv. in Appl. Math.* **30** (2003), 385–396.

[Wil2] N. J. Wildberger, "Minuscule posets from neighbourly graph sequences," *European J. Combin.* **24** (2003), 741-757.

[Wil3] N. J. Wildberger, "The mutation game, Coxeter graphs, and partially ordered multisets," preprint.