Edge-colored distributive lattices from representation theory

Rob Donnelly

Murray State University

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rob.donnelly@murraystate.edu

- Throughout these notes, annotations are boxed and in grey.

- Representation theory can mean different things to different people. The Calculus student might think of a definite integral representing an area under a curve. The analyst might think of trigonometric series representing a periodic function. And the algebraist might think of a collection of matrices representing some abstract algebraic structure such as a group or a Lie algebra.

Example of a representation: The symmetric group S_3

$$\varepsilon \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (13) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (123) \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(23) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad (132) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Question: Why are two of the group elements / matrices color coded?

  Answer: The two of them together generate the entire group when we multiply them together enough times.

- This suggests a principle: In representing an abstract algebraic structure, it can sometimes be enough to say where the generators go.

- BUT, we can't just randomly assign a matrix to a generator. The matrices representing generators must satisfy the same <u>relations</u> that the generators satisfy.

Example of a representation: the simple Lie algebra A_2

$$A_2 = \langle \, x_1, \, y_1, \, x_2, \, y_2 \; | \; \text{some relations} \, \rangle$$

- Lie algebras are vector spaces equipped with a certain nonassociative multiplication.
- The "simple" Lie algebras have beautiful presentations by generators and relations.

  For example, the simple Lie algebra A_2 above is generated by four distinguished elements, a red x-y pair and a blue x-y pair.
- Below, representing matrices for these generators are identified. They are 8×8 matrices, and are fairly sparse. The transposes of the "y" matrices are shown to facilitate a certain method of depicting such representations.

$$x_1 \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad x_2 \mapsto \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$y_1 \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \qquad y_2 \mapsto \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

"Adjacency matrix"

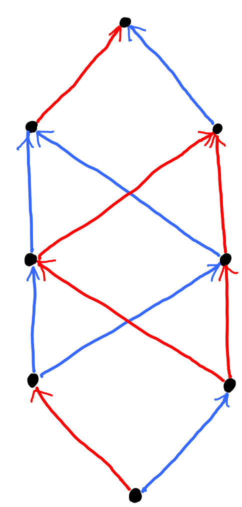
$$\begin{bmatrix} 0 & R & B & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B & B & 0 & 0 & 0 \\ 0 & 0 & 0 & R & R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B & R & 0 \\ 0 & 0 & 0 & 0 & 0 & B & R & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Stack the four representing matrices on top of each other (use the transposes of the y's) and note wherever there is a nonzero entry. Record the color of the matrix containing the nonzero entry.
- We thus get a sort-of adjacency matrix, from which we construct an edge-colored directed graph, as below.

A picture of this representation

8 vertices $\leftrightarrow$ 8 basis vectors

Directed colored edges $\leftrightarrow$ Actions of generators

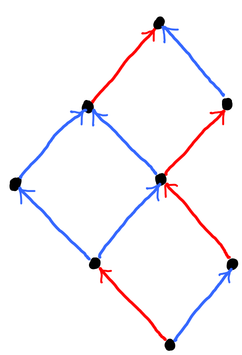


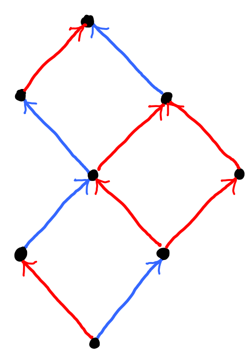
- One reason for looking at matrix representations of abstract algebraic structures is that sometimes we can take advantage of matrix algebra techniques, such as changing basis to get a different view of things.

Two changes of basis ...

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

... result in these two pictures ...



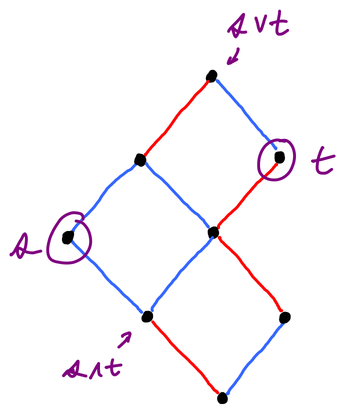


Both are order diagrams for distributive lattices —

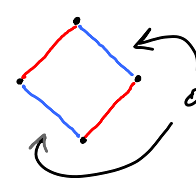
"diamond-colored" distributive lattices

- To the trained eye of the poset specialist, these pictures are order diagrams for special partially ordered sets known as distributive lattices. Next, I'll indicate what is meant by the terminology "diamond-colored distributive lattice."

Diamond-colored distributive lattices



- $\vee$ = least upper bound operation
- $\wedge$ = greatest lower bound operation
- $\vee$ distributes over $\wedge$ and vice-versa
- Diamond-colored ...



Opposite edges have the same color in any such "diamond"

- From here on, we quit writing the arrow tips on the directed edges, since it is to be understood that all edges point "up."
- It is obvious that in our picture above, any two vertices s and t have a least upper bound $s \vee t$ and a greatest lower bound $s \wedge t$. What is not so obvious is that lub "$\vee$" distributes over glb "$\wedge$" and vice-versa.
- Next we show a way to "compress" any diamond-colored distributive lattice into a smaller vertex-colored poset.

Compressing diamond-colored distributive lattices

- $x \in L$ is join irreducible if ...

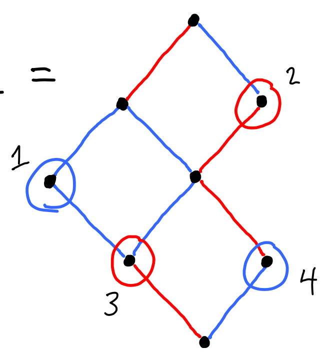
  there is precisely one $y \in L$ immediately below x:

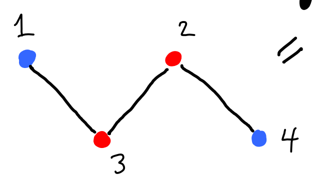
  (In this case give x the same color as the edge below)

- The vertex-colored induced order subposet of join irreducibles in L is denoted $\mathbf{j}_{\text{color}}(L)$.

$L =$



$\longrightarrow$



$= \mathbf{j}_{\text{color}}(L)$

- $y \in L$ is meet irreducible if ...

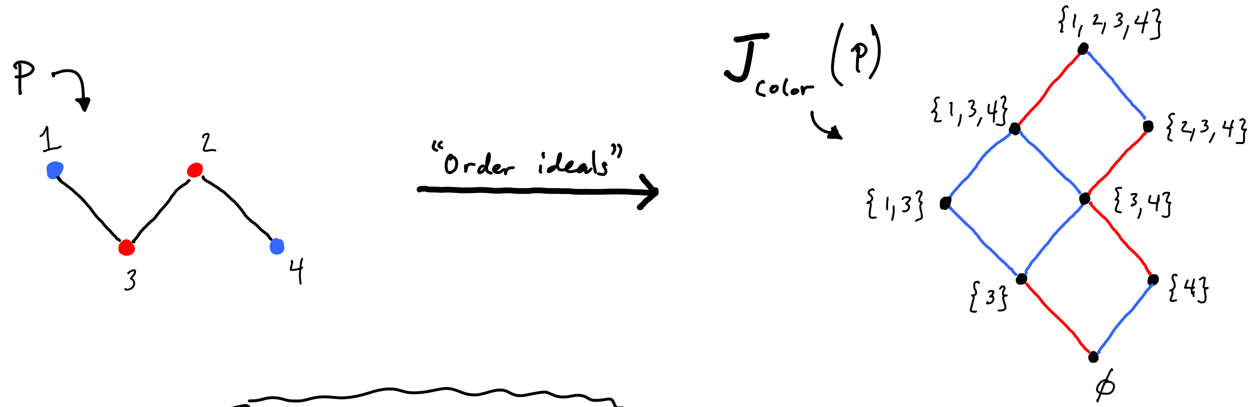
  there is precisely one $x \in L$ immediately above y:

  (In this case give y the same color as the edge above)

- The vertex-colored induced order subposet of meet irreducibles in L is denoted $\mathbf{m}_{\text{color}}(L)$.

Reconstructing diamond-colored distributive lattices from vertex-colored posets



Similarly, $M_{color}(P)$

- A compression technique is only good if you can reasonably recover the thing you compressed.
- To recover the diamond-colored distributive lattice from its vertex-colored poset of join irreducibles --- <u>First</u> look at all vertex subsets of P. There are 16. <u>Second</u>, keep only the "order ideals", those subsets that are "closed below." This means that if a subset contains vertex 2, then to be closed below the subset must also contain 3 and 4. There are 8 order ideals from P, and they have been organized into the lattice picture on the right. <u>Third</u>, there is an edge between order ideals if their set difference is one vertex. The edge gets the color of that vertex.
- The following theorem guarantees that big J and little j are in some sense inverses of one another.

Fundamental Theorem for Finite Diamond-colored Distributive Lattices

(1) L = diamond-colored distributive lattice

Then $L \cong \mathbf{J}_{\text{color}}\left(\mathbf{j}_{\text{color}}(L)\right) \cong \mathbf{M}_{\text{color}}\left(\mathbf{m}_{\text{color}}(L)\right)$

(2) P = vertex-colored poset

Then $P \cong \mathbf{j}_{\text{color}}\left(\mathbf{J}_{\text{color}}(P)\right) \cong \mathbf{m}_{\text{color}}\left(\mathbf{M}_{\text{color}}(P)\right)$

This generalizes a "monochromatic" theorem from the 1930's due to G. Birkhoff

- This theorem is a straightforward generalization of a classical "uncolored" result due to Birkhoff, sometimes called "Birkhoff's representation theorem" or the "Fundamental Theorem for Finite Distributive Lattices."

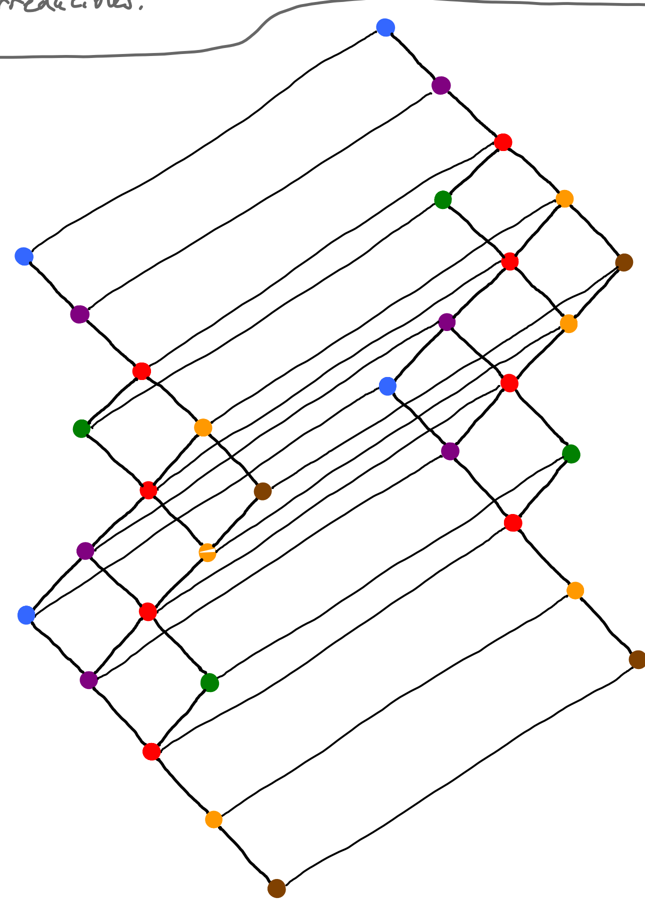
Where does this leave us?

1. Diamond-colored distributive lattice models for simple Lie algebra representations abound.

   The FTFDCDL gives a way to predict and/or nicely characterize these lattices by their posets of join irreducibles.

- Distributive lattice models for simple Lie algebra representations indeed abound, but they're not easy to find.
- Next is an example of a distributive lattice model for a representation of the exotic simple Lie algebra E_6. The model was found by predicting a certain pattern for its poset of join irreducibles.

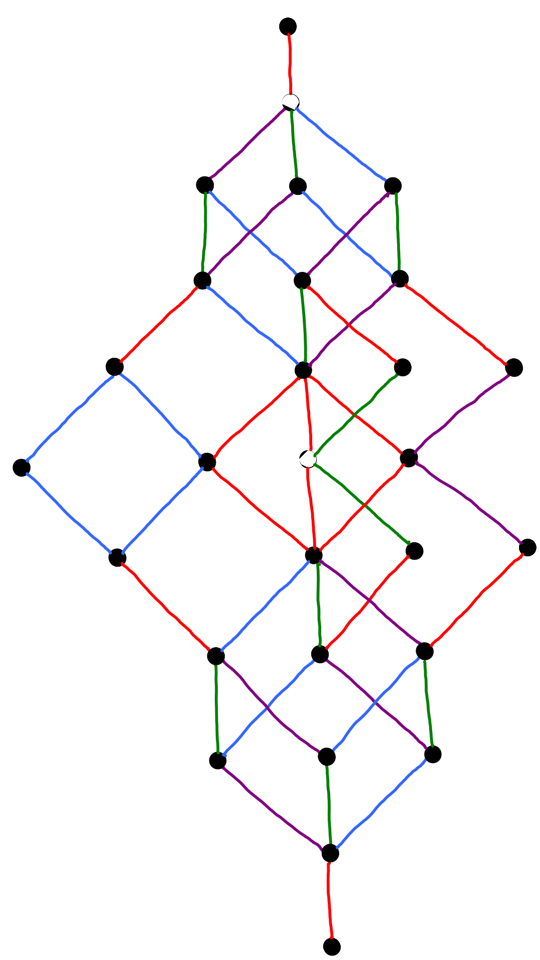
Poset of join irreducibles for a distributive lattice model for a 351-dimensional representation of E_6



2. Alas, not all lattice models for simple Lie algebra representations are distributive. ☹

Next, Josh Hyatt will talk about a more general lattice compression/reconstruction result we hope to apply to lattices like....

- In fact, this 28-dimensional representation for D_4 is the smallest irreducible representation of a simple Lie algebra which has no distributive lattice model.



Non-distributive lattice model for the 28-dimensional "adjoint" representation of D_4