Lie algebra representations
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Here's what I hope was accomplished in two prior talks:

- We defined a Lie algebra as a vector space $\mathfrak{g}$ over a ground field $\mathbb{F}$ (for us, $\mathbb{R}$ or $\mathbb{C}$) equipped with a bilinear, anticommutative, "Jacobi-associative" multiplication operation $[\, ,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

- We looked at many families of matrix Lie algebras, including the following:

  1. The general linear Lie algebras $\mathfrak{gl}(n, \mathbb{F})$ and $\mathfrak{gl}(V)$:

  $$\mathfrak{gl}(n, \mathbb{F}) = \{n \times n \text{ matrices } A \text{ over } \mathbb{F}\}, \text{ where } [A, B] := AB - BA$$

  $$\mathfrak{gl}(V) := \{\text{linear transformations } T : V \to V\}, \text{ where } [S, T] := ST - TS,$$

  where V is a vector space over the ground field $\mathbb{F}$.

  2. The special linear Lie algebra $\mathfrak{sl}(n, \mathbb{F})$ (a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$):

  $$\mathfrak{sl}(n, \mathbb{F}) = \{n \times n \text{ matrices } A \text{ over } \mathbb{F} \text{ with trace}(A) = 0\}$$

  3. The orthogonal Lie algebra $\mathfrak{so}(n, \mathbb{F})$ (a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$):

  $$\mathfrak{so}(n, \mathbb{F}) = \{A \in \mathfrak{gl}(n, \mathbb{F}) \,\big|\, A \text{ is skew-symmetric}\}$$

  4. The special unitary Lie algebra $\mathfrak{su}_n = \mathfrak{su}(n, \mathbb{C})$ (a real Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$):

  $$\mathfrak{su}(n, \mathbb{C}) = \{A \in \mathfrak{gl}(n, \mathbb{C}), \,\big|\, A \text{ is skew-Hermitian}\}$$

  We saw that $\mathfrak{su}(2, \mathbb{C}) \approx (\mathbb{R}^3, \times)$.

- We considered a class of complex Lie algebras defined by generators and relations, the Kac-Moody Lie algebras. We start with a "GCM graph" (Γ, A), where Γ is a finite simple graph and the matrix A is a "Generalized Cartan Matrix," a sort of adjacency matrix for the graph Γ. There are three generators (which span a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$) attached to each node of the graph Γ. The relations are determined by the the GCM graph — particularly the "intertwining" and "finiteness" relations — as illustrated in some examples below.

- In my talk at the Kentucky MAA meeting, I discussed a game played on GCM graphs and answered a finiteness question about this game. The answer is a classification by Dynkin diagrams, and leads to a classification of finite-dimensional Kac-Moody Lie algebras and finite Weyl groups. We will assume the following related result in this talk:

  Classification of Simple Lie Algebras (Cartan, Serre, Kac etc) *The finite-dimensional simple Lie algebras are precisely the finite-dimensional Kac-Moody Lie algebras $\mathfrak{g}(\Gamma, A)$ whose GCM graphs are connected Dynkin diagrams from our irredundant list.*

From generators and relations to concrete realizations:

Example 1 For the one-node GCM graph A_1 we have the Lie algebra

$$\mathfrak{g}(A_1) = \langle x, y, h \,\Big|\, [x, y] = h, [h, x] = 2x, [h, y] = -2y\rangle$$

As discussed last time, the Lie algebra homomorphism $\mathfrak{g}(A_1) \xrightarrow{\phi} \mathfrak{sl}(2,\mathbb{C})$ induced by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

yields an isomorphism of Lie algebras, so $\mathfrak{g}(A_1) \approx \mathfrak{sl}(2,\mathbb{C})$.

Example 2 For the two node GCM graph $A_2 =$ •→——←• we have the Lie algebra

$$\mathfrak{g}(A_2) = \langle x_1, y_1, h_1, x_2, y_2, h_2 \;\Big|\; \text{“Serre” relations} \,\rangle,$$

where the Serre relations in this case are:

“$\mathfrak{sl}(2,\mathbb{C})$” relations: $[x_1,y_1] = h_1$, $[h_1,x_1] = 2x_1$, $[h_1,y_1] = -2y_1$,
$[x_2,y_2] = h_2$, $[h_2,x_2] = 2x_2$, $[h_2,y_2] = -2y_2$.

“Commuting” relations: $[h_1,h_2] = 0$, $[x_1,y_2] = 0$, $[x_2,y_1] = 0$.

“Intertwining” relations: $[h_1,x_2] = -x_2$, $[h_1,y_2] = y_2$, $[h_2,x_1] = -x_1$, $[h_2,y_1] = y_1$

“Finiteness” relations: $[x_1,[x_1,x_2]] = [x_2,[x_2,x_1]] = [y_1,[y_1,y_2]] = [y_2,[y_2,y_1]] = 0$.

Consider the Lie algebra homomorphism $\mathfrak{g}(A_2) \xrightarrow{\phi} \mathfrak{gl}(3,\mathbb{C})$ induced by

$$x_1 \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y_1 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h_1 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad y_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad h_2 \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

To confirm that this indeed induces a Lie algebra homomorphism, check that the image matrices preserve the above relations. Since the image matrices all have trace zero, then $\mathrm{im}(\phi) \subseteq \mathfrak{sl}(3,\mathbb{C})$. In fact, in a prior talk we saw that these matrices generate all of $\mathfrak{sl}(3,\mathbb{C})$, so $\mathrm{im}(\phi) = \mathfrak{sl}(3,\mathbb{C})$. By the Classification of simple Lie algebras, since $\mathfrak{g}(A_2)$ is simple, then ϕ is injective, and therefore $\mathfrak{g}(A_2) \approx \mathfrak{sl}(3,\mathbb{C})$.

In a similar way, $\mathfrak{g}(A_n) \approx \mathfrak{sl}(n+1,\mathbb{C})$.

Example 3 For the two node GCM graph “B_2” $=$ •→→——←• (this is the same as C_2, but for now we'll write it this way) we have the Lie algebra

$$\mathfrak{g}(\text{“}B_2\text{”}) = \langle x_1, y_1, h_1, x_2, y_2, h_2 \;\Big|\; \text{“Serre” relations} \,\rangle,$$

where the Serre relations in this case are:

“$\mathfrak{sl}(2,\mathbb{C})$” relations: $[x_1,y_1] = h_1$, $[h_1,x_1] = 2x_1$, $[h_1,y_1] = -2y_1$,
$[x_2,y_2] = h_2$, $[h_2,x_2] = 2x_2$, $[h_2,y_2] = -2y_2$.

“Commuting” relations: $[h_1,h_2] = 0$, $[x_1,y_2] = 0$, $[x_2,y_1] = 0$.

“Intertwining” relations: $[h_1,x_2] = -x_2$, $[h_1,y_2] = y_2$, $[h_2,x_1] = -2x_1$, $[h_2,y_1] = 2y_1$

"Finiteness" relations: $[x_1,[x_1,x_2]] = [x_2,[x_2,[x_2,x_1]]] = [y_1,[y_1,y_2]] = [y_2,[y_2,[y_2,y_1]]] = 0$.

Consider the Lie algebra homomorphism $\mathfrak{g}(\text{“}B_2\text{”}) \xrightarrow{\phi} \mathfrak{gl}(5,\mathbb{C})$ induced by

$$x_1 \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad h_1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad h_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

To confirm that this indeed induces a Lie algebra homomorphism, check that the image matrices preserve the above relations. Since the image matrices all have trace zero, then $\text{im}(\phi) \subseteq \mathfrak{sl}(5,\mathbb{C})$. In fact, in a prior talk we saw that these matrices generate all of $\mathfrak{g}_{M'}$, where

$$M' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is congruent to $M = I$ over $\mathbb{C}$. Then we have $\text{im}(\phi) = \mathfrak{g}_{M'} \approx \mathfrak{so}(5,\mathbb{C})$. By the Classification of simple Lie algebras, since $\mathfrak{g}(\text{“}B_2\text{”})$ is simple, then ϕ is injective, and therefore $\mathfrak{g}(\text{“}B_2\text{”}) \approx \mathfrak{so}(5,\mathbb{C})$.

In a similar way, $\mathfrak{g}(B_n) \approx \mathfrak{so}(2n+1,\mathbb{C})$.

Example 4 A combinatorial representation of $\mathfrak{g}(A_1) \approx \mathfrak{sl}(2,\mathbb{C})$:

Let $\mathfrak{B}_n := \Big\{\text{subsets of } \{1,2,\ldots,n\}\Big\}$. Let $V = V[\mathfrak{B}_n] = \text{span}_{\mathbb{C}}\{v_S \,\big|\, S \in \mathfrak{B}_n\}$, a 2^n-dimensional complex vector space. Define linear transformations X, Y, and H on V as follows:

$$\begin{aligned} X(v_S) &:= \sum_{T\in\mathfrak{B}_n, T\supseteq S, |T\setminus S|=1} v_T \\ Y(v_S) &:= \sum_{R\in\mathfrak{B}_n, R\subseteq S, |S\setminus R|=1} v_R \\ H &:= [X,Y] \end{aligned}$$

Claim 1: $H(v_S) = (2|S| - n)v_S$

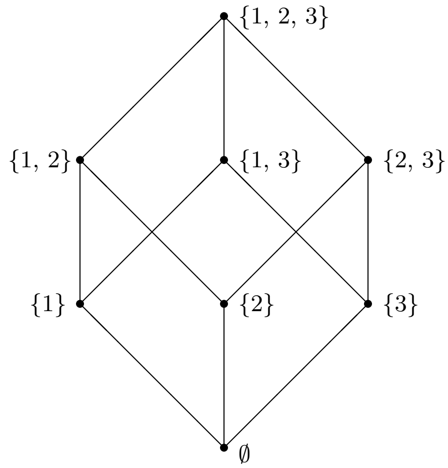
Claim 2: $[HX](v_S) = 2X(v_S)$

Claim 3: $[HY](v_S) = -2Y(v_S)$

So there is a Lie algebra homomorphism $\mathfrak{g}(A_1) \xrightarrow{\phi} \mathfrak{gl}(V)$ induced by $x \mapsto X$, $y \mapsto Y$, and $h \mapsto H$.

This can be visualized as follows: Regard $\mathfrak{B}_n$ to be a partially ordered set with respect to subset containment "$\subseteq$," that is, we have $S \subseteq T$ for $S, T \in \mathfrak{B}_n$ iff S is a subset of T when we think of S and T as subsets of $\{1, 2, \ldots, n\}$.

Below is a picture of $\mathfrak{B}_3$ with respect to this partial ordering. In this graph, an edge connects a subset S of $\{1, 2, 3\}$ to a subset T (with S below T) if $T \setminus S$ is a single element from $\{1, 2, 3\}$.



The "Boolean Lattice" $\mathfrak{B}_3$

We can view the action of $\mathfrak{g}(A_1)$ on $\mathfrak{B}_n$ in the following way: X takes the basis vector at a given vertex to the sum of the basis vectors above the given vertex; Y takes the basis vector at a given vertex to the sum of the basis vectors below the given vertex; and each basis vector is an eigenvector for H where the eigenvalue is "twice the rank of the vertex minus the length of the poset."

Representations and modules (i.e. Lie algebra actions):

- Although the following definitions work over arbitrary fields and vector spaces of any dimension, from here on our focus will be on finite-dimensional complex representations.
- A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism

  $$\phi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V) \quad (\text{or } \mathfrak{gl}(d, \mathbb{C})\,),$$

  where V is an $\mathbb{C}$-vector space of dimension d. For x in $\mathfrak{g}$ and $v \in V$, write $x.v := \phi(x)(v)$. Then:

  $$\begin{array}{lrcl} (1) & (ax+by).v & = & a(x.v)+b(y.v) \\ (2) & x.(av+bw) & = & a(x.v)+b(x.w) \\ (3) & [x,y].v & = & x.y.v - y.x.v \end{array}$$

  for all $x, y \in \mathfrak{g}$, $a, b \in \mathbb{C}$, and $v, w \in V$.
- If V is an $\mathbb{C}$-vector space of dimension d with an operation $\mathfrak{g} \times V \to V$ denoted $(x, v) \mapsto x.v$ and satisfying (1), (2), and (3) above, then we say V (together with the operation) is a *$\mathfrak{g}$-module*. In this case define $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ by the rule $\phi(x)(v) = x.v$ for all $x \in \mathfrak{g}$ and $v \in V$. Check that ϕ is a Lie algebra homomorphism.

So we see that representations of $\mathfrak{g}$ and $\mathfrak{g}$-modules are different language for the same phenomena. We'll use both in what follows.

- Suppose V and W are $\mathfrak{g}$-modules. Then we can create the following new $\mathfrak{g}$-modules:
  - $V \bigoplus W$ is a $\mathfrak{g}$-module via $x.(v,w) := (x.v, x.w)$.
  - $V \bigotimes W$ is a $\mathfrak{g}$-module via $x.(v \otimes w) := (x.v) \otimes w + v \otimes (x.w)$ (for simple tensors).
  - V^* is a $\mathfrak{g}$-module via $(x.f)(v) := -f(x.v)$, where $f : V \to \mathbb{F}$ is a linear functional in the dual space V^*.

- Suppose V is a $\mathfrak{g}$-module, and suppose W is a subspace of V such that $x.w \in W$ for all $x \in \mathfrak{g}$ and $w \in W$. Then we say that W is a <u>$\mathfrak{g}$-*stable*</u> subspace of V.

  The $\mathfrak{g}$-module V is <u>*irreducible*</u> if V has no $\mathfrak{g}$-stable subspaces other than $\{0\}$ and V.

  The $\mathfrak{g}$-module V is <u>*completely irreducible*</u> if for any $\mathfrak{g}$-stable subspace W of V there is a $\mathfrak{g}$-stable subspace W' of V so that $V = W \bigoplus W'$.

- We say V and W are <u>*isomorphic $\mathfrak{g}$-modules*</u> if there is a linear transformation $\psi : V \to W$ such that $\psi(x.v) = x.\psi(v)$ for all $x \in \mathfrak{g}$ and all $v \in V$.

<u>Complex semisimple Lie algebras:</u>

From here on our Lie algebras are complex. Continue with the assumption that representations are complex and finite-dimensional.

We say a Lie algebra $\mathfrak{g}$ is <u>*semisimple*</u> $\overset{\text{(def)}}{\Longleftrightarrow}$ $\mathfrak{g} \approx \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, where each $\mathfrak{g}_i$ is simple.

Observation *Let $\mathfrak{g}$ be semisimple, and let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Then $\phi(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$ (i.e. the image of $\mathfrak{g}$ is a collection of trace zero endomorphisms).*

Proof. If $\mathfrak{g}$ is simple then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Now check that if $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. If $z \in \mathfrak{g}$, then $z = \sum c_i[x_i, y_i]$. Then $\phi(z) = \sum c_i(\phi(x_i)\phi(y_i) - \phi(y_i)\phi(x_i)\,)$, and hence $\text{trace}(\phi(z)) = 0$. □

The following comments are intended to suggest why it is reasonable in studying Lie algebra representations to restrict our attention to semisimple Lie algebras.

We say a Lie algebra $\mathfrak{g}$ is <u>*solvable*</u> $\overset{\text{(def)}}{\Longleftrightarrow}$ the following sequence of subspaces is eventually the zero subspace: $\mathfrak{g}^{(0)} := \mathfrak{g}$, $\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^{(2)} := [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}], \ldots, \mathfrak{g}^{(i)} := [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$, etc. An abelian Lie algebra is solvable, and so is the matrix subalgebra $\{A \in \mathfrak{gl}(n, \mathbb{C}) \,\big|\, A \text{ is upper triangular}\}$.

Lie's Theorem *Suppose $\mathfrak{s}$ is a solvable Lie algebra, and suppose $\phi : \mathfrak{s} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{s}$ ($V \neq 0$). Then there exists a common eigenvector for the actions of all the elements $\phi(x)$ in $\phi(\mathfrak{s})$.*

Notice what this says about irreducible modules for solvable Lie algebras: they are all one-dimensional.

Levi's Theorem *Given a Lie algebra $\mathfrak{g}$, there exists a semisimple Lie subalgebra $\mathfrak{g}_{ss}$ and a solvable Lie subalgebra $\mathfrak{s}$ such that $\mathfrak{g} \approx \mathfrak{g}_{ss} \oplus \mathfrak{s}$. Moreover, if V is an irreducible $\mathfrak{g}$-module, then $V \approx W \bigotimes V_0$ (an isomorphism of $\mathfrak{g}$-modules), where W is an irreducible $\mathfrak{g}_{ss}$-module, V_0 is a (one-dimensional) irreducible $\mathfrak{s}$-module, the the action of $\mathfrak{g}_{ss}$ on V_0 is trivial, and the action of $\mathfrak{s}$ on W is trivial.*

Weyl's Theorem (Complete Reducibility) *If $\mathfrak{g}$ is semisimple and V is a $\mathfrak{g}$-module, then V is completely reducible. (Then V has a unique decomposition as a sum of irreducible $\mathfrak{g}$-modules.)*

Representations of complex semisimple Lie algebras:

We let $\mathfrak{g} = \mathfrak{g}(\Gamma, A)$ be a semisimple Lie algebra obtained by a generators-and-relations construction whose starting point is a GCM graph (Γ, A) whose connected components are connected Dynkin diagrams. Fix a numbering $1, 2, \ldots, n$ of the nodes of Γ. Let V be a $\mathfrak{g}$-module of finite dimension $d \geq 1$. Then:

1. Weight basis There exists a basis for V consisting of common eigenvectors with integer eigenvalues for all $\phi(h_i)$'s. That is, there is a basis $\{v_1, \ldots, v_d\}$ and integers $m_{i,j}$ such that $h_i.v_j = m_{i,j}v_j$ for all $1 \leq i \leq n$, $1 \leq j \leq d$.

   IDEA: Use basic $\mathfrak{sl}(2, \mathbb{C})$ theory to obtain a basis of eigenvectors with integral eigenvalues for each $\phi(h_i)$. Now the fact that the $\phi(h_i)$'s commute implies that a basis of common eigenvectors exists.

   DEFINITIONS: Such a basis for V is called a *weight basis*. A *weight vector* is any vector in V that is an eigenvector for all $\phi(h_i)$'s. The *weight* of a weight vector v is the n-tuple $(m_1, \ldots, m_n)$ of integral eigenvalues for which $\phi(h_i)(v) = m_i v$.

2. Maximal vector There is a weight vector v such that $x_i.v = 0$ for all x_i's. The weight of v is an n-tuple of *nonnegative* integers.

   IDEA: This follows from Lie's theorem applied to $\phi(\mathfrak{s})$, where $\mathfrak{s}$ is the solvable Lie subalgebra of $\mathfrak{g}$ generated by all of the x_i's and h_i's. That the eigenvalues are nonnegative follows from $\mathfrak{sl}(2, \mathbb{C})$ theory.

   DEFINITION: Any such vector is called a *maximal vector*.

3. Unique maximal vector $\leftrightarrow$ irreducible module V is irreducible if and only if there exists a unique maximal vector v (unique up to scalar factors).

   IDEA: Each maximal vector "generates" a subspace of V that is stable under the action of $\mathfrak{g}$.

   DEFINITION: If $\lambda = (\lambda_i)_{1 \leq i \leq n}$ is the weight of the maximal vector v, then we say the irreducible $\mathfrak{g}$-module V has *highest weight λ*.

4. Uniqueness Suppose V and W are irreducible $\mathfrak{g}$-modules with maximal vectors having the same highest weight. Then $V \approx W$.

   IDEA: Both modules are "cyclic" in the sense that they are generated by their maximal vectors. Since these maximal vectors are identical in the way $\mathfrak{g}$ acts on them, then the modules they generate should be the same.

5. Existence Suppose $\lambda = (\lambda_i)_{1 \leq i \leq n}$ is any n-tuple of nonnegative integers. Then there exists a finite-dimensional irreducible $\mathfrak{g}$-module V with highest weight λ.

   IDEA: Such a $\mathfrak{g}$-module can be obtained by generators and relations: For a suitable ideal $I(\lambda)$ in $U(\mathfrak{g})$, we have $V \approx U(\mathfrak{g})/I(\lambda)$.

   DEFINITION: A *dominant weight* is an n-tuple $\lambda = (\lambda_i)_{1 \leq i \leq n}$ of nonnegative integers.

Together #4 and #5 give a one-to-one correspondence between irreducible modules and dominant weights.