Lie algebras: definitions and examples
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- Definition A *Lie algebra* $(\mathfrak{g}, [,])$ is a vector space $\mathfrak{g}$ together with an operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is:
  1. Bilinear: $[x + \lambda y, z] = [x, z] + \lambda[y, z]$ and $[x, y + \lambda z] = [x, y] + \lambda[x, z]$
     for all $x, y, z \in \mathfrak{g}$ and for all scalars λ
  2. Anticommutative: $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$
  3. "Jacobi associative": $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$

- For us the ground field $\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$. The operation $[,]$ is called the *Lie bracket*. A *Lie subalgebra* of a Lie algebra is a subspace that is closed under the Lie bracket. An *ideal* is a Lie subalgebra that is "i-o closed." So a Lie subalgebra $\mathfrak{i}$ is an ideal if $[a, x] \in \mathfrak{i}$ for all $a \in \mathfrak{g}$ and $x \in \mathfrak{i}$. Notice that any ideal is automatically "two-sided" since $[a, x] \in \mathfrak{i}$ iff $[x, a] \in \mathfrak{i}$.

- Substructures, Homomorphisms, kernels, quotients, etc.

Algebraic structure	Group G	Ring R	Vector space W	Lie algebra $\mathfrak{g}$
Substructure	Subgroup H	Subring S	Subspace W	Lie subalgebra $\mathfrak{h}$
Homomorphism	$\phi(xy) = \phi(x)\phi(y)$	$\phi(x+y) = \phi(x)+\phi(y)$ $\phi(xy) = \phi(x)\phi(y)$	$\phi(x+y) = \phi(x)+\phi(y)$ $\phi(\lambda x) = \lambda\phi(x)$	$\phi(x+y) = \phi(x)+\phi(y)$ $\phi(\lambda x) = \lambda\phi(x)$ $\phi([x, y]) =$ $[\phi(x), \phi(y)]$
Kernel	Normal subgroup N	(Two-sided) ideal I	Subspace W	Ideal $\mathfrak{i}$
Quotient	G/N	R/I	V/W	$\mathfrak{g}/\mathfrak{i}$
Operations in quotient	$(xN)(yN) = (xy)N$	$(x + I) + (y + I) =$ $(x + y) + I$ $(x+I)(y+I) = xy+I$	$(x + W) + (y + W) =$ $(x + y) + W$ $\lambda(x+W) = (\lambda x)+W$	$(x + \mathfrak{i}) + (y + \mathfrak{i}) =$ $(x + y) + \mathfrak{i}$ $\lambda(x + \mathfrak{i}) = (\lambda x) + \mathfrak{i}$ $[(x + \mathfrak{i}), (y + \mathfrak{i})] =$ $[x, y] + \mathfrak{i}$

- Examples

1. Take any vector space $\mathfrak{g}$ over $\mathbb{F}$ and declare $[x, y] := 0$ for all $x, y \in \mathfrak{g}$. Then $(\mathfrak{g}, [,])$ is called an *abelian Lie algebra*. Notice that any one-dimensional Lie algebra is abelian.

2. $(\mathbb{R}^3, \times)$ is a real Lie algebra. See Stewart's *Calculus: Early Transcendentals*, 5th edition, §12.4, Theorem 8 and #43 (the latter is the Jacobi identity!).

3. The *general linear Lie algebra*: $\mathfrak{gl}(n, \mathbb{F}) := \{n \times n$ matrices with entries from $\mathbb{F}\}$ with Lie bracket $[A, B] := AB - BA$.

   Alternatively, take an n-dimensional vector space V over $\mathbb{F}$, and set $\mathfrak{gl}(V) := \{$endomorphisms $T : V \to V\}$ with Lie bracket $[S, T] := ST - TS$.

4. The *special linear Lie algebra*: $\mathfrak{sl}(n, \mathbb{F}) := \{$traceless $n \times n$ matrices with entries from $\mathbb{F}\}$.

Alternatively, take an n-dimensional vector space V over $\mathbb{F}$, and set $\mathfrak{sl}(V) := \{\text{endomorphisms } T : V \to V \text{ with zero trace}\}$.

Check that $\mathfrak{sl}(n, \mathbb{F})$ is an $(n^2 - 1)$-dimensional subspace of $\mathfrak{gl}(n, \mathbb{F})$, but it is not closed under matrix multiplication. However, $\mathfrak{sl}(n, \mathbb{F})$ is closed under the Lie bracket $[A, B] = AB - BA$ since $\text{trace}(AB) = \text{trace}(BA)$ and hence is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$.

The mapping trace : $\mathfrak{gl}(n, \mathbb{F}) \to \mathbb{F}$ is a surjective Lie algebra homomorphism. Moreover, $\mathfrak{sl}(n, \mathbb{F}) = \ker(\text{trace})$, and hence is an ideal in $\mathfrak{gl}(n, \mathbb{F})$. Finally, by the usual homomorphism theorems, it follows that the abelian Lie algebra $\mathbb{F}$ is isomorphic to the quotient $\mathfrak{gl}(n, \mathbb{F})/\mathfrak{sl}(n, \mathbb{F})$. (Thus it is easy to see that $\dim(\mathfrak{sl}(n, \mathbb{F})) = n^2 - 1$.)

5. Special cases: $n = 2$ and $n = 3$.

$\boxed{n = 2}$ The following matrices are a basis for the three-dimensional $\mathfrak{sl}(2, \mathbb{F})$:

$$x := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In fact, $\mathfrak{sl}(2, \mathbb{F}) = \langle x, y, h \,\big|\, [x, y] = h, [h, x] = 2x, [h, y] = -2y \rangle$ (generators and relations).

$\boxed{n = 3}$ The following matrices are a basis for the eight-dimensional $\mathfrak{sl}(3, \mathbb{F})$:

$$x_1 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y_1 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h_1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad y_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad h_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$[x_1, x_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad [y_2, y_1] := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In fact, $\mathfrak{sl}(3, \mathbb{F}) \approx \langle x_1, y_1, h_1, x_2, y_2, h_2 \,\big|\,$ "Serre" relations $\rangle$, where the Serre relations in this case are:

"$\mathfrak{sl}(2, \mathbb{F})$" relations: $[x_1, y_1] = h_1$, $[h_1, x_1] = 2x_1$, $[h_1, y_1] = -2y_1$,
$[x_2, y_2] = h_2$, $[h_2, x_2] = 2x_2$, $[h_2, y_2] = -2y_2$.

"Commuting" relations: $[h_1, h_2] = 0$, $[x_1, y_2] = 0$, $[x_2, y_1] = 0$.

"Intertwining" relations: $[h_1, x_2] = -x_2$, $[h_1, y_2] = y_2$, $[h_2, x_1] = -x_1$, $[h_2, y_1] = y_1$

"Finiteness" relations: $[x_1, [x_1, x_2]] = [x_2, [x_2, x_1]] = [y_1, [y_1, y_2]] = [y_2, [y_2, y_1]] = 0$.

6. Combinatorial representation/realization of $\mathfrak{sl}(2, \mathbb{F})$.

7. *Lie subalgebras associated to bilinear forms*:

Let $M \in \mathfrak{gl}(n, \mathbb{F})$. Think of M as a matrix representative of some bilinear form. Set

$$\mathfrak{g}_M := \{A \in \mathfrak{gl}(n, \mathbb{F}) \,\big|\, A^T M + MA = O\}.$$

Then $\mathfrak{g}_M$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$.

On conjugate versions of $\mathfrak{g}_M$ If $P \in GL(n,\mathbb{F})$ and $M' := P^T M P$, then $\mathfrak{g}_{M'} \approx \mathfrak{g}_M$. For the mapping $\phi : \mathfrak{gl}(n,\mathbb{F}) \to \mathfrak{gl}(n,\mathbb{F})$ given by $\phi(X) = P^{-1}XP$, check that $\phi\Big|_{\mathfrak{g}_M} : \mathfrak{g}_M \to \mathfrak{g}_{M'}$ is an isomorphism of Lie algebras. So $\mathfrak{g}_{M'} = P^{-1}\mathfrak{g}_M P$.

On symmetry/skew-symmetry In what follows we will focus on symmetric and skew-symmetric choices for M. This is actually a fairly reasonable assumption to make. We have the following direct sum of vector spaces: $\mathfrak{gl}(n,\mathbb{F}) = Symm \bigoplus Skew$, where $Symm$ is the subspace of symmetric matrices and $Skew$ is the subspace of skew-symmetric matrices. Note that for any matrix X, $X = \frac{1}{2}(X+X^T)+\frac{1}{2}(X-X^T)$. Let $X_{symm} := \frac{1}{2}(X+X^T)$, and let $X_{skew} := \frac{1}{2}(X-X^T)$. Then $\mathfrak{g}_M = \mathfrak{g}_{M_{symm}} \cap \mathfrak{g}_{M_{skew}}$.

On nondegeneracy If we think of M as a matrix representative of a *nondegenerate* bilinear form, then $M \in GL(n,\mathbb{F}) = \{\text{invertible } n \times n \text{ matrices with entries from } \mathbb{F}\}$. In this case $\mathfrak{g}_M$ is a Lie subalgebra of $\mathfrak{sl}(n,\mathbb{F})$ since $A^TM + MA = O$ iff $M^{-1}A^TM + A = O$, hence $0 = \text{trace}(M^{-1}A^TM + A) = \text{trace}(M^{-1}A^TM) + \text{trace}(A) = \text{trace}(A^T) + \text{trace}(A) = 2\text{trace}(A)$. Now suppose a symmetric or skew-symmetric M is degenerate with rank $r < n$. Then there is a matrix congruent to M which has the block form $\begin{pmatrix} \tilde{M} & O \\ O & O \end{pmatrix}$ with $\tilde{M} \in GL(r,\mathbb{F})$ and $\tilde{M}$ symmetric or skew-symmetric. If we write $A = \begin{pmatrix} \tilde{A} & B \\ C & D \end{pmatrix}$, we have $A^TM + MA = O$ iff $\tilde{A} \in \mathfrak{g}_{\tilde{M}}$ and $B = O$, in which case we can freely choose C and D. So in studying the Lie algebra $\mathfrak{g}_M$ we'll end up studying the Lie subalgebra $\mathfrak{g}_{\tilde{M}}$ for the nondegenerate $\tilde{M}$ anyway. I think that $\mathfrak{a} := \left\{\begin{pmatrix} O & O \\ C & O \end{pmatrix}\right\}$ is an abelian ideal in $\mathfrak{g}_M$ and that $\mathfrak{g}_M/\mathfrak{a} \approx \mathfrak{g}_{\tilde{M}} \oplus \mathfrak{gl}(n-r,\mathbb{F})$.

8. Special cases of Lie algebras associated to bilinear forms:

   (a) Take $M = I$. Think of M as a matrix representing a symmetric positive definite bilinear form. Then $\mathfrak{g}_M =$ "$\mathfrak{so}(n,\mathbb{F})$" $= \{\text{skew-symmetric } n \times n \text{ matrices}\}$. These are the *orthogonal Lie algebras*. Special case: $n = 5$. Notice that the only diagonal matrix that is skew-symmetric is the zero matrix. There is another matrix representation of $\mathfrak{so}(5,\mathbb{C})$ that has some nontrivial diagonal matrices. Identifying such matrices will be of crucial importance when we discuss representations of complex simple Lie algebras. Now the matrix

$$M' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   is congruent to $M = I$ over $\mathbb{C}$. That is, for some $P \in GL(5,\mathbb{C})$ we have $M' = P^TMP = P^TIP = P^TP$. Then $\mathfrak{g}_{M'} \approx \mathfrak{so}(5,\mathbb{C})$. A basis for $\mathfrak{g}_{M'}$ will therefore have $\frac{n^2-n}{2} = \frac{5^2-5}{2} = 10$ basis vectors.

$$x_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad h_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Plus these four vectors:

$$[x_2, x_1], [x_2, [x_2, x_1]], [y_2, y_1], \text{ and } [y_2, [y_2, y_1]].$$

In fact, $\mathfrak{so}(5, \mathbb{C}) \approx \langle x_1, y_1, h_1, x_2, y_2, h_2 \mid \text{"Serre" relations} \rangle$, where the Serre relations in this case are:

"$\mathfrak{sl}(2, \mathbb{F})$" relations: $[x_1, y_1] = h_1$, $[h_1, x_1] = 2x_1$, $[h_1, y_1] = -2y_1$,
$[x_2, y_2] = h_2$, $[h_2, x_2] = 2x_2$, $[h_2, y_2] = -2y_2$.

"Commuting" relations: $[h_1, h_2] = 0$, $[x_1, y_2] = 0$, $[x_2, y_1] = 0$.

"Intertwining" relations: $[h_1, x_2] = -x_2$, $[h_1, y_2] = y_2$, $[h_2, x_1] = -2x_1$, $[h_2, y_1] = 2y_1$

"Finiteness" relations: $[x_1, [x_1, x_2]] = [x_2, [x_2, [x_2, x_1]]] = [y_1, [y_1, y_2]] = [y_2, [y_2, [y_2, y_1]]] = 0$.

(b) Take $M =$ "$M_{p,q}$" $:= \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix}$. Think of M as a matrix representing a nondegenerate symmetric indefinite bilinear form.

Then $\mathfrak{g}_M =$ "$\mathfrak{so}(p, q, \mathbb{F})$." When $\mathbb{F} = \mathbb{C}$, M is congruent to I, i.e. $M = P^T P$ for some matrix $P \in GL(p+q, \mathbb{C})$. In this case, $\mathfrak{so}(p, q, \mathbb{C}) \approx \mathfrak{so}(p+q, \mathbb{C})$. When $\mathbb{F} = \mathbb{R}$, the $\mathfrak{so}(p, q, \mathbb{R})$'s are called *pseudo-orthogonal Lie algebras*.

(c) Take $M = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}$. Think of M as a matrix representing a nondegenerate skew-symmetric bilinear form. Even dimensionality is a requirement for a skew-symmetric bilinear form to be nondegenerate.

Then $\mathfrak{g}_M =$ "$\mathfrak{sp}(2n, \mathbb{F})$." These are the *symplectic Lie algebras*.

9. The *unitary* and *special unitary* Lie algebras:

Set $\mathfrak{u}_n := \{A \in \mathfrak{gl}(n, \mathbb{C}) \mid A^* + A = O\}$. Here A^* means conjugate transpose. Thus $\mathfrak{u}_n$ consists of the *skew-Hermitian* complex matrices. Set $\mathfrak{su}_n := \{A \in \mathfrak{sl}(n, \mathbb{C}) \mid A^* + A = O\}$.

These are <u>real</u> (NOT complex) Lie algebras: For $\lambda \in \mathbb{C}$ and $A \in \mathfrak{gl}(n, \mathbb{C})$, see that $(\lambda A)^* = \overline{\lambda} A^*$. If $\lambda A \in \mathfrak{u}_n$ or $\mathfrak{su}_n$, then $(\lambda A)^* = -\lambda A$. We have $-\overline{\lambda} A = -\lambda A$ for nonzero A only if λ is real.

<u>Special case</u>: $n = 2$. Check that $A \in \mathfrak{su}_2$ if and only if $A = \begin{pmatrix} ai & b + ci \\ -b + ci & -ai \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$. Thus $\dim_{\mathbb{R}}(\mathfrak{su}_2) = 3$. The mapping $\phi : \mathbb{R}^3 \to \mathfrak{su}_2$ given by

$$\phi(x, y, z) = \begin{pmatrix} \frac{1}{2}xi & \frac{1}{2}(y + zi) \\ \frac{1}{2}(-y + zi) & \frac{1}{2}(-xi) \end{pmatrix}$$

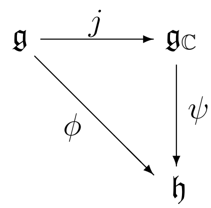
is an isomorphism of Lie algebras, so $(\mathbb{R}^3, \times) \approx \mathfrak{su}_2$.

"*Pseudo-unitary Lie algebras*" have a relationship to the matrix $M_{p,q} = \begin{pmatrix} I_p & O \\ O & -I_q \end{pmatrix}$ similar to the pseudo-orthogonal Lie algebras: $\mathfrak{u}_{p,q} := \{A \in \mathfrak{gl}(n, \mathbb{C}) \mid A^* M_{p,q} + M_{p,q} A = O\}$ and $\mathfrak{su}_{p,q} := \{A \in \mathfrak{sl}(n, \mathbb{C}) \mid A^* M_{p,q} + M_{p,q} A = O\}$

What about the skew-Hermitian case? Is there something like a symplectic unitary Lie algebra? It turns out that this offers nothing new. Suppose M is nondegenerate skew-Hermitian ($M^* = -M$). Then M is *normal* (i.e. $M^*M = MM^*$), and by a linear algebra theorem there exists a unitary matrix P (i.e. $P^{-1} = P^*$) such that P^*MP is diagonal. Then we can find a diagonal matrix $Q \in GL(n, \mathbb{C})$ so that $D = Q^*P^*MPQ$ is diagonal with diagonal entries of modulus 1. Since D is also skew-Hermitian, then its diagonal entries are purely imaginary, so $D = iM_{p,q}$ for some nonnegative integers p and q ($p + q = n$). Now observe that $\mathfrak{g}_M \approx \mathfrak{g}_D \approx \mathfrak{g}_{M_{p,q}}$.

- *Complexification*:

  Let $\mathfrak{g}$ be a real Lie algebra. Then there exists a unique pair $(\mathfrak{g}_\mathbb{C}, j)$ such that $\mathfrak{g}_\mathbb{C}$ is a complex Lie algebra, such that $j : \mathfrak{g} \to \mathfrak{g}_\mathbb{C}$ is a homomorphism of real Lie algebras (j will turn out to be injective), and such that we have the following "universal" property: Whenever $\mathfrak{h}$ is a complex Lie algebra and $\phi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of real Lie algebras, there is a unique homomorphism $\psi : \mathfrak{g}_\mathbb{C} \to \mathfrak{h}$ of complex Lie algebras which makes the following diagram commute:



  That is, $\phi = \psi \circ j$. The complex Lie algebra $\mathfrak{g}_\mathbb{C}$ can be realized by extending scalars on $\mathfrak{g}$: If we think of $\mathfrak{g}$ as a real vector space, then we can give the complex vector space $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ a Lie bracket operation that naturally extends the Lie bracket for $\mathfrak{g}$.

  In practice, for a finite-dimensional real Lie algebra $\mathfrak{g}$ with basis $\{v_1, \ldots, v_d\}$, then $\mathfrak{g}_\mathbb{C}$ is the complex vector space with basis $\{v_1, \ldots, v_d\}$. Then for $x, y \in \mathfrak{g}_\mathbb{C}$, we have

$$[x, y] = \left[\sum a_i v_i, \sum b_j w_j\right] := \sum a_i b_j [v_i, v_j],$$

  where each $[v_i, v_j]$ is calculated in $\mathfrak{g}$ and expressed as a real linear combination in the basis $\{v_k\}$. It should be apparent now that $\mathbb{R}_\mathbb{C} = \mathbb{C}$, that $\mathfrak{sl}(2, \mathbb{R})_\mathbb{C} = \mathfrak{sl}(2, \mathbb{C})$, etc.

- What it means to be *simple*:

  An algebraic structure $\mathcal{A}$ is *simple* if its kernels are always trivial or all of $\mathcal{A}$. That is, $\mathcal{A}$ is simple if for any nontrivial homomorphism $\phi : \mathcal{A} \to \mathcal{A}'$, it is the case that $\ker(\phi)$ is trivial. Recall that $\ker(\phi)$ is trivial iff ϕ is injective, so $\mathcal{A}$ is simple iff all of its nontrivial homomorphic images are isomorphic to $\mathcal{A}$.

  One of the great (apparent) achievements of 20th century mathematics is the classification of finite simple groups. These are: the alternating groups on ≥ 5 letters, the cyclic groups of prime order, the finite simple groups of Lie type, and the 26 "sporadic" finite simple groups, which includes the MONSTER, a simple group of order

$$808, 017, 424, 794, 512, 875, 886, 459, 904, 961, 710, 757, 005, 754, 368, 000, 000, 000.$$

  One can easily confirm that a commutative ring with unity is simple iff it is a field. A finite field must have order p^n for some prime p, and any two fields with the same finite order are isomorphic. One can construct a finite field of any given prime power order. This amounts then to a classification of finite simple commutative rings with unity.

- Some examples and nonexamples of simple Lie algebras:

  $\boxed{\mathfrak{gl}(n,\mathbb{F}) \text{ is not simple}}$ This is because $\mathfrak{sl}(n,\mathbb{F})$ is a proper nontrivial ideal. (See Example #4 above.)

  $\boxed{\mathfrak{sl}(2,\mathbb{F}) \text{ is simple}}$ Together the following two observations show that any nontrivial ideal $\mathfrak{i}$ in $\mathfrak{sl}(2,\mathbb{F})$ must be all of $\mathfrak{sl}(2,\mathbb{F})$:

  Observation 1: If an ideal $\mathfrak{i}$ in $\mathfrak{sl}(2,\mathbb{F})$ contains any one of the generators x, y, or h, then $\mathfrak{i}$ contains all of the generators x, y, and h, and hence $\mathfrak{i} = \mathfrak{sl}(2,\mathbb{F})$. This is easy, since for example $x \in \mathfrak{i}$ implies that $[x,y] = h \in \mathfrak{i}$ and thus $-\frac{1}{2}[h,y] = y \in \mathfrak{i}$.

  Observation 2: If $ax+by+ch \neq 0$ is in $\mathfrak{i}$, then $[[ax+by+ch,x],x] = -2bx$ and $[[ax+by+ch],y],y] = -2ay$ are in $\mathfrak{i}$. So if $a \neq 0$ or $b \neq 0$, then $\mathfrak{i} = \mathfrak{sl}(2,\mathbb{F})$ by Observation 1.

  $\boxed{\mathfrak{sl}(3,\mathbb{F}) \text{ is simple}}$ We'll outline an argument for this one, again using observations based on calculations with the generating elements.

  Observation 1′: If an ideal $\mathfrak{i}$ in $\mathfrak{sl}(3,\mathbb{F})$ contains any one of x_i, y_i, or h_i $(i = 1,2)$ then $\mathfrak{i}$ contains all of them. To see this, note that for $j \in \{1,2\}$, it follows from Observation 1 that if one of x_j, y_j, h_j is in $\mathfrak{i}$, then $\{x_j, y_j, h_j\} \subset \mathfrak{i}$. If $h_1 \in \mathfrak{i}$, then $[h_1, y_2] = y_2 \in \mathfrak{i}$, and if $h_2 \in \mathfrak{i}$, then $[h_2, y_1] = y_1 \in \mathfrak{i}$. So the intertwining relations allow us to show that $\{x_1, y_1, h_1\} \subset \mathfrak{i}$ iff $\{x_2, y_2, h_2\} \subset \mathfrak{i}$.

  Observation 2′: Now suppose $a_1x_1 + b_1y_1c_1h_1 + a_1x_1 + b_1y_1c_1h_1 + p[x_1,x_2] + q[y_2,y_1] \neq 0$ is in $\mathfrak{i}$. To finish the argument use calculations similar to those of Observation 2 above to show that $\mathfrak{i}$ must contain at least one of the generators x_i, y_i, or h_i $(i = 1,2)$.

- Simple Lie algebras:

  The following table exhibits some infinite families of finite-dimensional real simple Lie algebras. There are three other infinite families of finite-dimensional real simple Lie algebras which can be obtained by looking at analogs over the quaternions $\mathbb{H}$ of $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{so}(p,q,\mathbb{R})$, and $\mathfrak{sp}(2n,\mathbb{R})$. (This accounts for all of the infinite families.)

Real simple Lie algebra	Complexification
$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{C})$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C}) \times \mathfrak{sl}(n,\mathbb{C})$
$\mathfrak{so}(n,\mathbb{R})$	$\mathfrak{so}(n,\mathbb{C})$
$\mathfrak{so}(p,q,\mathbb{R})$	$\mathfrak{so}(p+q,\mathbb{C})$
$\mathfrak{so}(n,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C}) \times \mathfrak{so}(n,\mathbb{C})$
$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{sp}(2n,\mathbb{C})$
$\mathfrak{sp}(2n,\mathbb{C})$	$\mathfrak{sp}(2n,\mathbb{C}) \times \mathfrak{sp}(2n,\mathbb{C})$
$\mathfrak{su}_n$	$\mathfrak{sl}(n,\mathbb{C})$
$\mathfrak{su}_{p,q}$	$\mathfrak{sl}(p+q,\mathbb{C})$

  The following tables present the complete irredundant list of finite-dimensional complex simple Lie algebras.

Complex simple Lie algebra	Dimension
$A_n = \mathfrak{sl}(n+1,\mathbb{C})$, $n \geq 1$	$n^2 + 2n$
$B_n = \mathfrak{so}(2n+1,\mathbb{C})$, $n \geq 2$	$2n^2 + n$
$C_n = \mathfrak{sp}(2n,\mathbb{C})$, $n \geq 3$	$2n^2 + n$
$D_n = \mathfrak{so}(2n,\mathbb{C})$, $n \geq 4$	$2n^2 - n$

Complex simple Lie algebra	Dimension
E_6	78
E_7	133
E_8	248
F_4	52
G_2	14