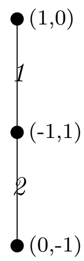
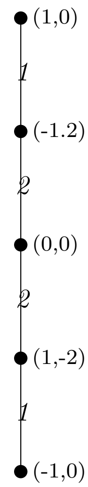
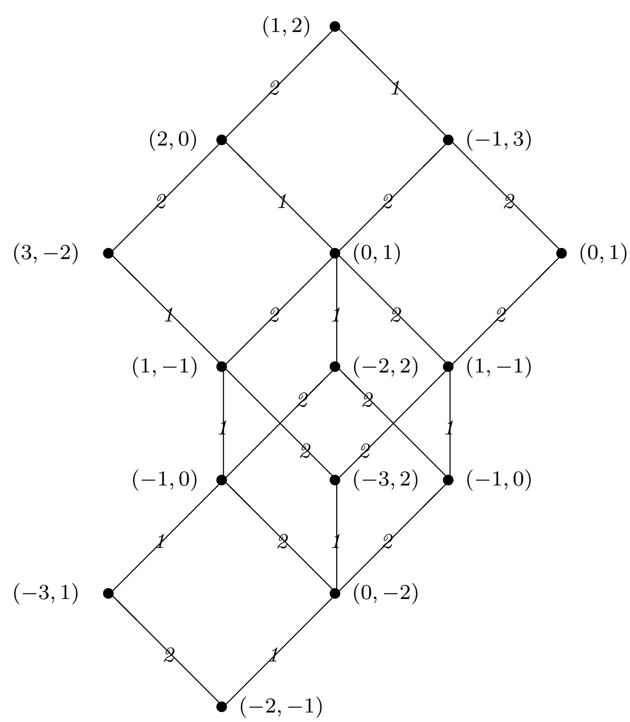
**Combinatorial models for Lie algebra representations: Pictures**
**Rob Donnelly April 14, 2006**

Pictured on the left is the supporting graph for the matrix representation of $\mathfrak{g}(A_2)$ given in Example 2 of the April 7, 2006 talk; on the right is the supporting graph for the matrix representation of $\mathfrak{g}(``B_2")$ given in Example 3. The integer pairs $(m_1(\mathbf{s}), m_2(\mathbf{s}))$ associated with the nodes in each graph are the weights (pairs of eigenvalues) for the corresponding basis vectors. (In this and other figures, edges are directed "up.")

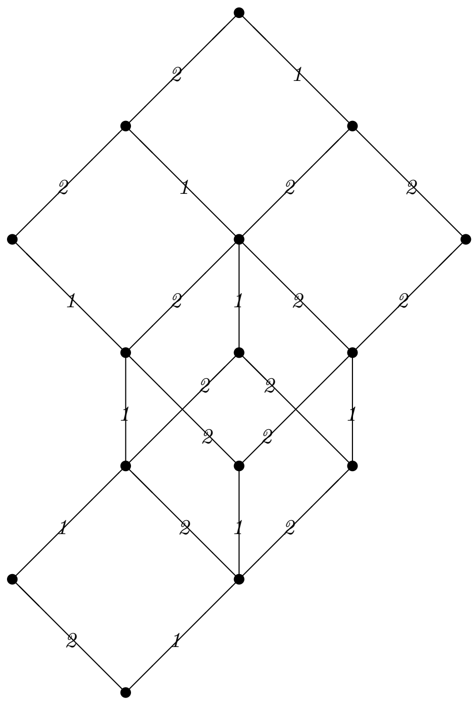



Pictured below is a supporting graph for a representation of $\mathfrak{g}(A_2)$. As above, the integer pairs $(m_1(\mathbf{s}), m_2(\mathbf{s}))$ at the nodes of this graph are weights. The representation is irreducible with highest weight $\lambda = (a, b) = (1, 2)$.



The following supporting graph $R$ for $\mathfrak{g}(A_2)$ is rank symmetric and rank unimodal.



$6^{\text{th}}$ rank; $|\rho^{-1}(6)| = 1$

$5^{\text{th}}$ rank; $|\rho^{-1}(5)| = 2$

$4^{\text{th}}$ rank; $|\rho^{-1}(4)| = 3$

$3^{\text{rd}}$ rank; $|\rho^{-1}(3)| = 3$

$2^{\text{nd}}$ rank; $|\rho^{-1}(2)| = 3$

$1^{\text{st}}$ rank; $|\rho^{-1}(1)| = 2$

$0^{\text{th}}$ rank; $|\rho^{-1}(0)| = 1$

The dimension formula works out as follows:

$$\text{card}(R) \overset{\text{Theorem}}{=\!=} \frac{(a+1)(b+1)(a+b+2)}{2} = \frac{(2)(3)(5)}{2} = 15$$

The rank generating function formula works out as follows:

$$\begin{aligned}
\text{rgf}(R,q) := \sum_{\mathbf{s}\in R} q^{\text{rank}(\mathbf{s})} &\overset{\text{Theorem}}{=\!=} \frac{(1-q^{a+1})(1-q^{b+1})(1-q^{a+b+2})}{(1-q)(1-q)(1-q^2)} \\
&= \frac{(1-q^2)(1-q^3)(1-q^5)}{(1-q)(1-q)(1-q^2)} \\
&= (1+q+q^2)(1+q+q^2+q^3+q^4) \\
&= 1+2q+3q^2+3q^3+3q^4+2q^5+q^6
\end{aligned}$$

The character formula works out as follows:

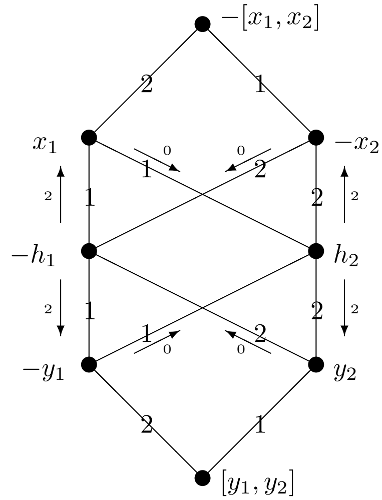
$$\text{char}(R) := \sum_{\mathbf{s}\in R} x^{m_1(\mathbf{s})} y^{m_2(\mathbf{s})} \overset{\text{Theorem}}{=\!=}$$

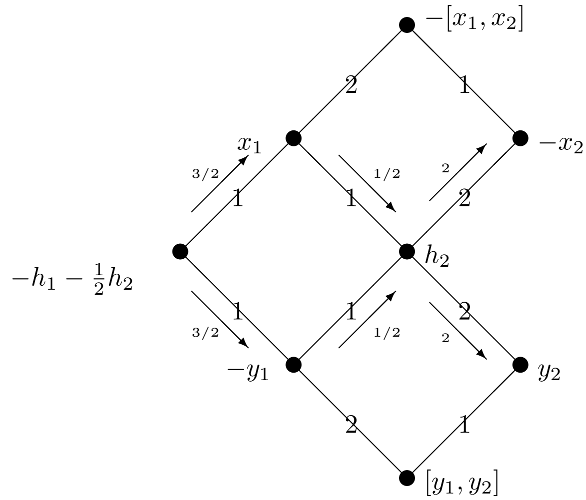
$$\begin{aligned}
&= \frac{x^{a+1}y^{b+1} - x^{-(a+1)}y^{a+b+2} - x^{a+b+2}y^{-(b+1)} + x^{-(a+b+2)}y^{a+1} + x^{b+1}y^{-(a+b+2)} - x^{-(b+1)}y^{-(a+1)}}{xy(1-x^{-2}y)(1-xy^{-2})(1-x^{-1}y^{-1})} \\
&= \frac{x^2y^3 - x^{-2}y^5 - x^5y^{-3} + x^{-5}y^2 + x^3y^{-5} - x^{-3}y^{-2}}{xy(1-x^{-2}y)(1-xy^{-2})(1-x^{-1}y^{-1})} \quad \text{(try this on Maple)} \\
&= xy^2 + x^{-1}y^3 + x^2 + 2y + x^3y^{-2} + x^{-2}y^2 + 2xy^{-1} + x^{-3}y^2 + 2x^{-1} + y^{-2} + x^{-3}y + x^{-2}y^{-1}
\end{aligned}$$

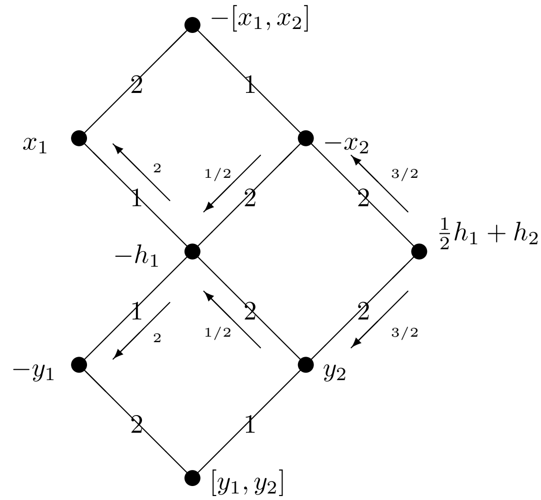
Three bases for the adjoint representation of $\mathfrak{g}(A_2) \approx \mathfrak{sl}(3, \mathbb{C})$.

For the adjoint representation, the Lie algebra acts on itself.

The Lie algebra homomorphism $\mathrm{ad} : \mathfrak{g}(A_2) \to \mathfrak{gl}(\mathfrak{g}(A_2)\,)$ is given by $\mathrm{ad}(z)(w) = [z, w]$.

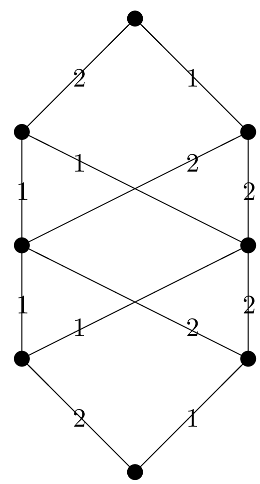


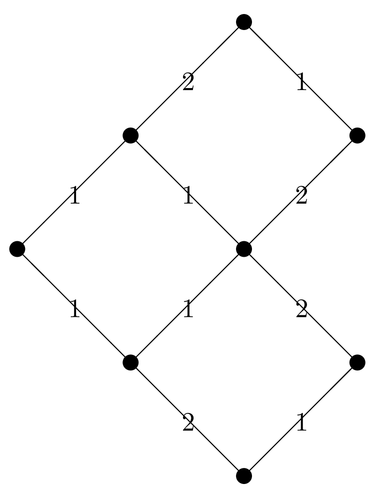



The <u>only</u> three supporting for the adjoint representation of $\mathfrak{g}(A_2) \approx \mathfrak{sl}(3, \mathbb{C})$.

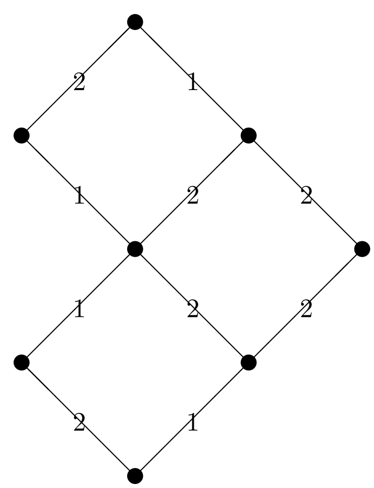
The “maximal” support



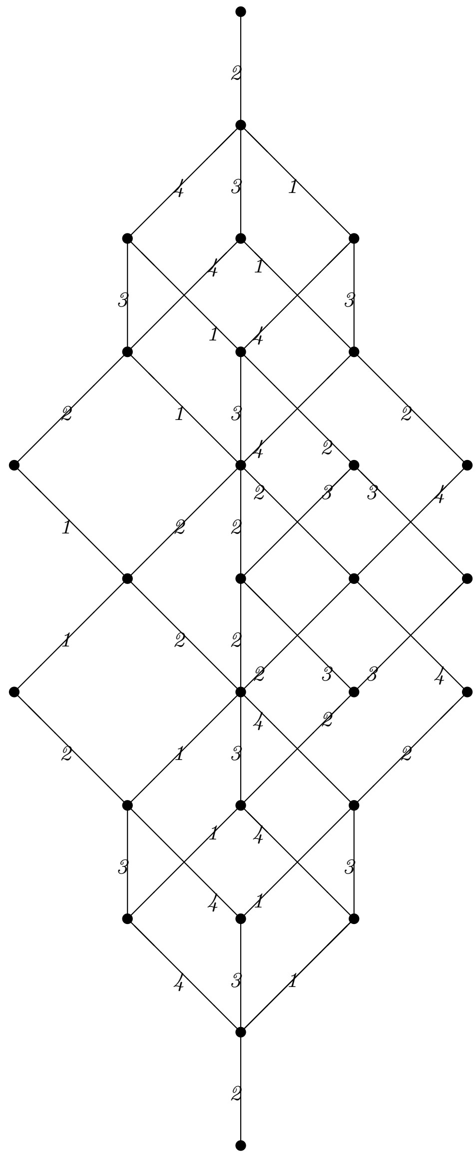

An “extremal” support



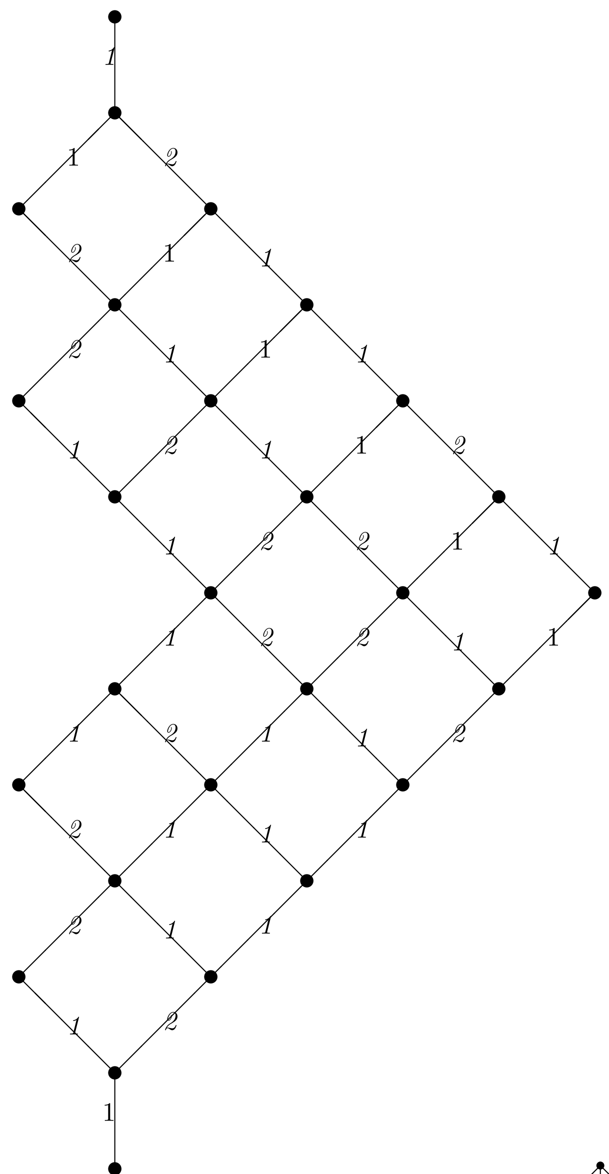

An “extremal” support



Pictured below is a supporting graph for a "fundamental" representation of the even orthogonal algebra $\mathfrak{so}(8, \mathbb{C}) \approx \mathfrak{g}(D_4)$



Pictured below is a supporting graph for an irreducible representation of a semisimple Lie algebra $\mathfrak{g}(\Gamma, A)$. Can you identify the alegbra?



By the way, do you know a nice formula for the rank generating function for the Boolean lattices $\mathfrak{B}_n$?