Coxeter groups and Combinatorics

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Unit 3: Asymmetric Geometric Representations of Coxeter Groups — PART 2

Throughout these notes, "References" come from the annoted references webpage linked to from the site where these notes are posted.

We will show how there are many ways to view an arbitrary Coxeter group as a collection of invertible linear transformations on a real vector space whose geometry is given by a possibly asymmetric bilinear form. These representations were discovered independently by Vinberg (1970's) and Eriksson (1990's). One object of interest for us will be a convex cone (the so-called Tits cone, named after Jacques Tits, a Belgian/French mathematician, Abel prize winner in 2008, and progenitor of much of the basic theory of Coxeter groups) created by an associated action of the Coxeter group on a certain "polyhedral" fundamental domain. We will connect these Coxeter group representations/actions to the numbers game of Unit 1.

- ### The contragredient of the geometric representation

→ $\mathcal{G} = (\Gamma, A)$ an SC graph with nodes $\{\gamma_i\}_{i \in I_n}$ indexed by an n-set I_n.

Let $W := W(\mathcal{G})$ be the Coxeter group $\langle \mathcal{S} \mid \mathcal{R} \rangle$,

where $\mathcal{S} = \{ s_i \}_{i \in I_n}$ and $\mathcal{R} = \{ (s_i s_j)^{m_{ij}} = \varepsilon \}_{i,j \in I_n}$

with $$m_{ij} = \begin{cases} 1 & \text{if } i = j \\ k_{ij} & \text{if } i \neq j \text{ and } a_{ij} a_{ji} = 4\cos^2(\pi/k_{ij}) \text{ with } k_{ij} \in \{2, 3, 4, \dots\} \\ \infty & \text{if } i \neq j \text{ and } a_{ij} a_{ji} \geq 4 \end{cases}$$

We have $V = \mathrm{span}_{\mathbb{R}}\{\alpha_i\}_{i\in I_n}$ with bilinear form $B : V\times V \to \mathbb{R}$ given by $B(\alpha_i, \alpha_j) = \frac{1}{2} a_{ij}$.

Let $S_i : V \to V$ be given by $S_i(v) = v - 2B(\alpha_i, v)\,\alpha_i$, for each $i \in I_n$.

Then the geometric representation $\sigma = \sigma_A$ is the injective homomorphism for which $\sigma(s_i) = S_i$.

→ Consider the contragredient representation $\sigma^* : W \to GL(V^*)$, necessarily injective as well. Use $\mathcal{B}^* = \{\omega_i\}_{i\in I_n}$ to denote the basis for V^* dual to the basis $\mathcal{B} = \{\alpha_i\}_{i\in I_n}$ for V. So, $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$.

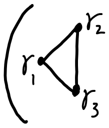
Let $\lambda = \sum \lambda_i \omega_i \in V^*$. Write $\lambda^{NEW} := s_i . \lambda$, with $\lambda^{NEW} = \sum \lambda_j^{NEW} \omega_j$.

Then relative to σ^*,

$$\begin{aligned}
\lambda_j^{NEW} = \langle \lambda^{NEW}, \alpha_j \rangle &= \langle s_i . \lambda, \alpha_j \rangle \\
&= \langle \lambda, s_i . \alpha_j \rangle \\
&= \langle \lambda, \alpha_j - a_{ij}\alpha_i \rangle \\
&= \langle \lambda, \alpha_j \rangle - a_{ij}\langle \lambda, \alpha_i \rangle \\
&= \langle \textstyle\sum \lambda_k \omega_k, \alpha_j \rangle - a_{ij}\langle \textstyle\sum \lambda_k \omega_k, \alpha_i \rangle \\
&= \textstyle\sum \lambda_k \langle \omega_k, \alpha_j \rangle - a_{ij} \textstyle\sum \lambda_k \langle \omega_k, \alpha_i \rangle \\
&= \lambda_j - a_{ij}\lambda_i
\end{aligned}$$

So, the numbers game is a combinatorial model for σ^*

In other words, $\lambda^{NEW} := s_i . \lambda$ can be viewed as the result of firing node γ_i from position λ on $\mathcal{D}$.

$\boxed{\text{Exercise}}$ For a generic three-node SC-graph $\mathcal{G} = \Big($ 
$, \begin{bmatrix} 2 & a_{12} & a_{13} \\ a_{21} & 2 & a_{23} \\ a_{31} & a_{32} & 2 \end{bmatrix} \Big)$, write down $[\sigma^*(s_i)]_{\mathcal{B}^*}$ for $i = 1, 2, 3$.

- ## The Tits Cone

Let $\mathcal{D} = \{ \lambda \in V^* \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \quad \forall\, i \in I_n \}$. In other words, $\mathcal{D}$ consists of the dominant positions for $\mathcal{G}$. Call $\mathcal{D}$ the "dominant chamber".

Let $C := \operatorname{int}(\mathcal{D}) = \{ \lambda \in V^* \mid \langle \lambda, \alpha_i^\vee \rangle > 0 \quad \forall\, i \in I_n \}$.

<u>Definition</u> The Tits Cone is the set $\mathcal{U} := \bigcup_{w \in W} w(\mathcal{D})$.

That is, $\mathcal{U} = \{ w.\lambda \mid w \in W, \lambda \in \mathcal{D} \}$.

<u>FACT</u>: $\mathcal{U}$ is a convex cone in the sense that if $\mu, \lambda \in \mathcal{U}$, then $(1-t)\mu + t\lambda \in \mathcal{U}$ for $0 \leq t \leq 1$.

[Reference: See [B4], § 5.13 That proof for the standard geometric representation also goes through in our asymmetric setting.]

<u>Eriksson's Tits Cone Convergence Theorem</u> Reference: See [P16]

$-\mathcal{U} = \{ \lambda \in V^* \mid$ there is a convergent game sequence from start position $\lambda \}$.

Before we prove the Theorem, first a Corollary.

__Corollary__ W is finite $\Longleftrightarrow$ $\mathcal{U} = V^*$

__Proof "$\Rightarrow$"__ With W finite, let $\lambda \in V^*$. Let p be the length of w_0, the longest element of W.

Let $(\gamma_{j_1}, \ldots, \gamma_{j_q})$ be a legal firing sequence from λ.

Eriksson's Reduced Word Theorem from Unit 3, Part 1

Since $s_{i_q} \cdots s_{i_1}$ is reduced, then $q \leq p$. In particular, from λ there is a convergent game sequence. Hence $\lambda \in -\mathcal{U}$. So $V^* \subseteq -\mathcal{U}$, which forces $V^* = -\mathcal{U} = \mathcal{U}$.

__Proof "$\Leftarrow$"__ If $\mathcal{U} = V^*$, then $-\mathcal{U} = V^*$ as well. Take $\lambda = (1, \ldots, 1)$. Then for any reduced expression $s_{i_p} \cdots s_{i_1}$, the firing sequence $(\gamma_{i_1}, \ldots, \gamma_{i_p})$ is legal. Since $\lambda \in -\mathcal{U}$, then there is a convergent game sequence from λ, and by strong convergence all game sequences from λ converge to the same position in the same number of moves. In particular, no reduced expression can have length exceeding the length of any game sequence played from λ. Since there is an upper bound on the lengths of group elements, W must be finite. //

By partial converse to Eriksson's Reduced Word Theorem from Unit 3, Part 1

We need some lemmas before we prove Eriksson's Tits Cone Convergence Theorem.

The next lemma is a translation of the Fundamental Theorem for Geometric Representations in the environment of the contragredient action.

<u>Lemma 1</u> Let $A_i = \{ f \in V^* \mid \langle f, \alpha_i \rangle > 0 \}$, so then

$\overline{A_i} = \{ f \in V^* \mid \langle f, \alpha_i \rangle \geq 0 \}$, $-A_i = \{ f \in V^* \mid \langle f, \alpha_i \rangle < 0 \}$,

and $\overline{-A_i} = \{ f \in V^* \mid \langle f, \alpha_i \rangle \leq 0 \}$. Let $i \in I_n$ and $w \in W$.

Then, $\ell(s_i w) < \ell(w) \Rightarrow w(C) \subseteq -A_i$ and $w(D) \subseteq \overline{-A_i}$.

Also, $\ell(s_i w) > \ell(w) \Rightarrow w(C) \subseteq A_i$ and $w(D) \subseteq \overline{A_i}$.

<u>Proof:</u> We'll only show $\ell(s_i w) < \ell(w) \Rightarrow w(D) \subseteq \overline{-A_i}$, as the proofs of the other claims of the lemma statement are entirely similar.

Say $g = w.f$ for some $f \in D$. Then

$$\langle g, \alpha_i \rangle = \langle w.f, \alpha_i \rangle = \langle f, w^{-1}.\alpha_i \rangle$$

Now $\ell(s_i w) < \ell(w) \Rightarrow \ell(w^{-1} s_i) < \ell(w^{-1})$.

So, $w^{-1}.\alpha_i \in \Phi^-$ by the FTGR. Then,

$$\langle f, w^{-1}.\alpha_i \rangle = \langle f, \sum_j c_j \alpha_j \rangle \qquad (c_j \leq 0)$$

$$= \sum_j \underbrace{c_j}_{\substack{\leq 0, \text{ since} \\ \sum c_j \alpha_j \in \Phi^-}} \underbrace{\langle f, \alpha_j \rangle}_{\geq 0, \text{ since } f \in D}$$

$$\leq 0$$

So, $\langle g, \alpha_i \rangle \leq 0$, hence $g \in \overline{-A_i}$. //

<u>Lemma 2</u> Let $\mu \in D$, and set $J := \{ j \in I_n \mid \mu_j = 0 \}$.

Then $w.\mu = \mu \iff w \in W_J$. Also, if $\mu = w.\mu'$ for some $\mu' \in D$ and $w \in W$, then $\mu = \mu'$.

<u>Proof:</u> First, we show that $w.\mu = \mu \iff w \in W_J$.

($\Leftarrow$) For any $\nu \in D$, if $\nu_j = 0$ then $s_j.\nu = \nu$.
To see this, note that for all $k \in I_n$,

$$\begin{aligned}
\langle s_j.\nu, \alpha_k \rangle &= \langle \nu, s_j.\alpha_k \rangle \\
&= \langle \nu, \alpha_k - a_{jk}\alpha_j \rangle \\
&= \langle \nu, \alpha_k \rangle - a_{jk}\langle \nu, \alpha_j \rangle \\
&= \langle \nu, \alpha_k \rangle - \cancelto{0}{a_{jk}\nu_j} \\
&= \langle \nu, \alpha_k \rangle
\end{aligned}$$

Therefore $s_j.\nu = \nu$. It follows that $w.\mu = \mu$ for all $w \in W_J$.

($\Rightarrow$) Induct on $\ell(w)$. If $\ell(w) = 0$ with $w.\mu = \mu$, then $w = \varepsilon \in W_J$.
Now suppose that for some non-negative integer k, it is the case that for all $w' \in W$ with $w'.\mu = \mu$ and $\ell(w') \leq k$, then $w \in W_J$. Then take $w \in W$ with $\ell(w) = k+1$ and $w.\mu = \mu$. Since $w \neq \varepsilon$ ($\ell(w) > 0$), then there is some $j \in I_n$ such that $\ell(s_j w) < \ell(w)$. Then by Lemma 1, $w(D) \subseteq \overline{-A_j}$. In particular, $w.\mu = \mu = \overline{-A_j}$.
So $\mu_j \leq 0$. But $\mu \in D \Rightarrow \mu_j \geq 0$, hence $\mu_j = 0$. So $j \in J$.
Then $(s_j w).\mu = s_j.(w.\mu) = s_j.\mu = \mu$. So $s_j w \in W_J$ by the induction hypothesis. It follows that $w \in W_J$.

For the 2nd part, suppose $\mu = w.\mu'$ for some $\mu' \in D$ and $w \in W$. We use induction on $\ell(w)$ to show that $\mu = \mu'$. If $\ell(w) = 0$, the result is clear. So now suppose that for some non-negative integer k, it is the case that whenever $\mu = w.\mu'$ with $\mu' \in D$ and $\ell(w) \leq k$, then $\mu = \mu'$. Suppose now that $\mu = w.\mu'$ for some $\mu' \in D$ with $\ell(w) = k+1$.

Then, let $j \in I_n$ such that $\ell(s_j w) < \ell(w)$. Then $s_j w.\mu' \in -\overline{A_j}$ means that $\mu_j \leq 0$. Since $\mu_j \geq 0$ as well ($\mu \in D$), then $\mu_j = 0$. But then $s_j w.\mu' = s_j.\mu = \mu$. By the induction hypothesis, since $\ell(s_j w) < \ell(w)$, then $\mu' = \mu$. //

Proof of Eriksson's Tits Cone Convergence Theorem

First, suppose there is a convergent game sequence from some $\lambda \in V^*$, say $(\gamma_{i_1}, \ldots, \gamma_{i_p})$. Then the terminal position $\mu := s_{i_p} \cdots s_{i_1}.\lambda \in -D$. That is, $s_{i_1} \cdots s_{i_p}.\mu = \lambda$, so $\lambda \in -\mathcal{U}$.

Second, for any $\lambda \in -\mathcal{U}$, let $\mathrm{LENGTH}(\lambda)$ be the length of any shortest $w \in W$ for which $\lambda = w.\mu$ for some $\mu \in -D$. We induct on $\mathrm{LENGTH}(\lambda)$ to show that there is a convergent game sequence from λ. In fact we will prove something stronger:

That all game sequences played from start position λ converge to some fixed $\mu \in -D$ in LENGTH(λ) steps; moreover that if $\lambda = w.\mu'$ for some $\mu' \in -D$, then $\mu' = \mu$; moreover, that there is a unique shortest $w \in W$ for which $\lambda = w.\mu$; that $(\gamma_{i_1}, \ldots, \gamma_{i_k})$ is a game sequence from λ if and only if $s_{i_1} \cdots s_{i_k}$ is a reduced expression for this shortest w; and that this shortest $w \in W^J$ where $J = \{ j \in I_n \mid \mu_j = 0 \}$.

If LENGTH$(\lambda) = 0$, then $\lambda \in -D$, so all games played from λ have length 0. Clearly the unique shortest w is $w = \varepsilon$, which is a member of W^J by definition. Of the remaining claims, the only one that is not obvious is that if $\lambda = w.\mu'$ for some $\mu' \in -D$, it must be the case that $\mu' = \lambda$. But this actually follows from Lemma 2.

Now suppose that for some non-negative integer k, the above claims hold whenever LENGTH$(\lambda') \leq k$ for $\lambda' \in -\mathcal{U}$.

Take $\lambda \in -\mathcal{U}$ with LENGTH$(\lambda) = k+1$. Write $\lambda = w.\mu$ with $\ell(w) = k+1$ and $\mu \in -D$. Pick any $i \in I_n$ so that $\ell(s_i w) < \ell(w)$. Then by Lemma 1 above, $w(-D) \subseteq \overline{A_i}$, so $\lambda \in \overline{A_i}$. Then either

$$\langle \lambda, \alpha_i \rangle = 0 \qquad \text{or} \qquad \langle \lambda, \alpha_i \rangle > 0 .$$

If $\langle \lambda, \alpha_i \rangle = 0$, then $\langle s_i . \lambda, \alpha_j \rangle = \langle \lambda, s_i . \alpha_j \rangle$

$= \langle \lambda, \alpha_j - a_{ij} \alpha_i \rangle = \langle \lambda, \alpha_j \rangle - a_{ij} \cdot \overset{0}{\cancel{\langle \lambda, \alpha_i \rangle}}$

$= \langle \lambda, \alpha_j \rangle$, for each $j \in I_n$.

In particular, $(s_i w).\mu = s_i.(w.\mu) = s_i.\lambda = \lambda$.

But this contradicts the fact that w is a <u>shortest</u> element of W for which λ can be written $w.\mu$: $\lambda = (s_i w).\mu$ and $s_i w$ is shorter than w !

Therefore, $\langle \lambda, \alpha_i \rangle > 0$. Then, firing λ at node γ_i is legal.

Let $\lambda^{NEW} := s_i . \lambda = (s_i w).\mu$. Clearly $\lambda^{NEW} \in -\mathcal{U}$.

Then, LENGTH$(\lambda^{NEW}) \leq k$, and the induction hypothesis applies.

Write $\lambda^{NEW} = v.\mu'$ for some shortest possible $v \in W$ and $\mu' \in -D$.

Suppose $k' := \ell(v) < k$. Then take a game sequence $(\gamma_{i_1}, \ldots, \gamma_{i_{k'}})$ from position λ^{NEW} (one exists by the induction hypothesis). Then, $s_{i_{k'}} \cdots s_{i_1} . \lambda^{NEW} \in -D$, so $s_{i_{k'}} \cdots s_{i_1} s_i . \lambda \in -D$. Again, this contradicts the fact that LENGTH$(\lambda) = k+1$. So, we must have $k' = \ell(v) = k$.

So the game sequence $(\gamma_{i_1}, \ldots, \gamma_{i_{k'=k}})$ from λ^{NEW} has length k.

Thus, $(\gamma_i, \gamma_{i_1}, \ldots, \gamma_{i_k})$ is a game sequence from position λ of length $k+1$.

By Strong Convergence, all game sequences from λ have length $k+1$, and converge to μ'.

So we have $\lambda = w.\mu$. Suppose $\lambda = u.\nu$ for some $\nu \in -D$ and $u \in W$. Then $u^{-1}w.\mu = \nu$, and by Lemma 2 it follows that $\mu = \nu$. So in particular, since $s_{i_{k+1}} \cdots s_{i_1} s_i . \lambda = \mu'$, then $\lambda = s_i s_{i_1} \cdots s_{i_k} . \mu'$, and hence $\mu' = \mu$. So all game sequences played from λ converge to μ.

Now say $\lambda = w'.\mu$ for some other shortest w'. Then $w'.\mu = w.\mu \Rightarrow$ $(w')^{-1}w.\mu = \mu$. Hence $(w')^{-1}w \in W_J$. Write $w = w^J w_J$ and $w' = (w')^J (w')_J$. The fact that each of w and w' are shortest means that we must have $w \in W^J$ and $w' \in W^J$, hence $w_J = (w')_J = \varepsilon$. Then $(w')^{-1}w \in W_J \Rightarrow w = w'v$ for some $v \in W_J$. Uniqueness of the factors w^J and w_J $\Rightarrow$ $v = w_J = \varepsilon$ and $w' = w^J = w$. Then $w' = w$, and $w \in W^J$.

Say $(\gamma_{i_1}, \ldots, \gamma_{i_{k+1}})$ is a game sequence from λ. Then $s_{i_{k+1}} \cdots s_{i_1} . \lambda = \mu$, and since $s_{i_{k+1}} \cdots s_{i_1}$ is shortest, then by previous paragraph, $w = s_{i_1} \cdots s_{i_{k+1}}$. Clearly this is reduced. Now suppose $s_{i_1} \cdots s_{i_{k+1}}$ is any reduced expression for w. Reasoning as in the middle paragraph of page 9 shows that we can take any $i_1 \in I_n$ for which $\ell(s_{i_1} w) < \ell(w)$ and from this build the game sequence $(\gamma_{i_1}, \ldots, \gamma_{i_{k+1}})$ to be played from λ. This completes the induction step, and the proof. /

Corollary (of the proof)

Let $\lambda \in -\mathcal{U}$. Then there is a unique $\mu \in -D$ for which $\lambda = w.\mu$ for some $w \in W$. Moreover, there is a unique shortest $w \in W$ for which $\lambda = w.\mu$. For this shortest w, we have $w \in W^J$, where $J = \{ j \in I_n \mid \mu_j = 0 \}$.
Any game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots)$ played from λ has length $\ell(w)$ and converges to position μ. For any reduced expression $s_{i_1} \cdots s_{i_k}$ for w, $(\gamma_{i_1}, \ldots, \gamma_{i_k})$ is a game sequence.

- **Example** Consider $\mathcal{G} = \gamma_1 \bullet \xrightarrow{\;p\;} \xleftarrow{\;q\;} \bullet\, \gamma_2$, with $pq = 4$.

Then $W = W(\mathcal{G}) = D_\infty$, the infinite dihedral group. What's the Tits cone here?

Relative to the basis $\mathcal{B}^* = \{\omega_1, \omega_2\}$ for V^*, we have

$$\underline{X} := [\sigma^*(s_1)]_{\mathcal{B}^*} = \begin{bmatrix} -1 & 0 \\ p & 1 \end{bmatrix} \quad \text{and} \quad \underline{Y} := [\sigma^*(s_2)]_{\mathcal{B}^*} = \begin{bmatrix} 1 & q \\ 0 & -1 \end{bmatrix}.$$

Check that $$\underline{Y}\,\underline{X} = \begin{bmatrix} 3 & q \\ -p & -1 \end{bmatrix} = \underbrace{\frac{1}{p}\begin{bmatrix} -1 & 2 \\ p & -p \end{bmatrix}}_{\underline{P}} \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{\underline{T}} \underbrace{\begin{bmatrix} p & 2 \\ p & 1 \end{bmatrix}}_{\underline{P}^{-1}}$$

$$\Rightarrow \quad (\underline{Y}\,\underline{X})^k = \frac{1}{p}\begin{bmatrix} -1 & 2 \\ p & -p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} p & 2 \\ p & 1 \end{bmatrix} \quad \text{for all } k \geq 1$$

Then: $(\mathbb{Y}\mathbb{X})^k = \begin{bmatrix} 2k+1 & 4k/p \\ -kp & -(2k-1) \end{bmatrix}$, $\mathbb{X}(\mathbb{Y}\mathbb{X})^k = \begin{bmatrix} -(2k+1) & -4k/p \\ (k+1)\cdot p & 2k+1 \end{bmatrix}$

$(\mathbb{Y}\mathbb{X})^k\mathbb{Y} = \begin{bmatrix} 2k+1 & 4(k+1)/p \\ -kp & -(2k+1) \end{bmatrix}$, $\mathbb{X}(\mathbb{Y}\mathbb{X})^k\mathbb{Y} = \begin{bmatrix} -(2k+1) & -4(k+1)/p \\ (k+1)p & 2k+3 \end{bmatrix}$

(All these formulas work for $k \geq 0$.)

At $p=q=2$, this gives

$(\mathbb{Y}\mathbb{X})^k = \begin{bmatrix} 2k+1 & 2k \\ -2k & -2k+1 \end{bmatrix}$ $\qquad \mathbb{X}(\mathbb{Y}\mathbb{X})^k = \begin{bmatrix} -2k-1 & -2k \\ 2k+2 & 2k+1 \end{bmatrix}$

$(\mathbb{Y}\mathbb{X})^k\mathbb{Y} = \begin{bmatrix} 2k+1 & 2k+2 \\ -2k & -2k-1 \end{bmatrix}$ $\qquad \mathbb{X}(\mathbb{Y}\mathbb{X})^k\mathbb{Y} = \begin{bmatrix} -2k-1 & -2k-2 \\ 2k+2 & 2k+3 \end{bmatrix}$

Then for a dominant position $\lambda = a\omega_1 + b\omega_2 = \begin{bmatrix} a \\ b \end{bmatrix}$ ($a \geq 0$, $b \geq 0$, not both zero):

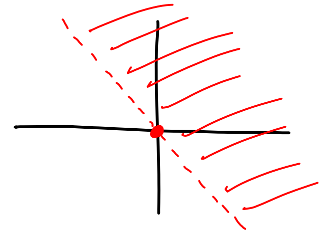
$(\mathbb{Y}\mathbb{X})^k \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (2k+1)a + 2kb \\ -2ka - 2kb + b \end{bmatrix} = \begin{bmatrix} z_1 \\ -z_1 + a + b \end{bmatrix}$, where $z_1 := (2k+1)a + 2kb$

$\mathbb{X}(\mathbb{Y}\mathbb{X})^k \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2ka - a - 2kb \\ (2k+2)a + (2k+1)b \end{bmatrix} = \begin{bmatrix} -z_1 \\ z_1 + a + b \end{bmatrix}$, " " "

$$Y X^k Y \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (2k+1)a + (2k+2)b \\ -2ka - 2kb - b \end{bmatrix} = \begin{bmatrix} z_2 \\ -z_2 + a + b \end{bmatrix}, \text{ where } z_2 := (2k+1)a + (2k+2)b$$

$$X (Y X)^k Y \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2ka - a - 2kb - 2b \\ (2k+2)a + (2k+3)b \end{bmatrix} = \begin{bmatrix} -z_2 \\ z_2 + a + b \end{bmatrix}, \quad \text{''} \quad \text{''} \quad \text{''}$$

It follows that $\mathcal{U} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; \begin{matrix} y > -x \\ \text{or } x = y = 0 \end{matrix} \right\}$



<u>Some notes about this example...</u>

→ In Davis' book, Appendix D, he has $\mathcal{U} = \{ \begin{bmatrix} x \\ y \end{bmatrix} \mid y \geq -x \}$, which (Reference: See [B3]) appears to be incorrect.

→ $\boxed{\text{Exercise}}$ For any positive p, q with $pq = 4$, show that $\mathcal{U} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; \begin{matrix} y > -\frac{p}{2}x \text{ or} \\ x = y = 0 \end{matrix} \right\}$.

→ Observe that $W = D_\infty$ is infinite here and that $\mathcal{U} \cap (-\mathcal{U}) = \{0\}$

(Of course, for finite W we have shown that $\mathcal{U} = V^* = -\mathcal{U}$.)

- <u>Revisiting our main finiteness question for the Numbers Game</u>

<u>Definition</u> A connected SC-graph $\mathcal{G}$ is admissible if it has a nontrivial dominant position from which there is a convergent game sequence.

That is, the connected graph $\mathcal{G}$ is admissible $\iff$ $D \cap (-\mathcal{U}) \neq \{0\}$.

Def (cont'd) A connected SC-graph $\mathcal{G}$ is inadmissible if it is not admissible, i.e. $D \cap (-\mathcal{U}) = \{0\}$.

Observation For any SC graph, $D \cap (-\mathcal{U}) = \{0\} \iff \mathcal{U} \cap (-\mathcal{U}) = \{0\}$.

Proof: ($\Leftarrow$) is obvious, so we only prove ($\Rightarrow$). Now $\lambda \in \mathcal{U} \cap (-\mathcal{U})$ means $\lambda = w_1 \cdot \mu_1 = -w_2 \cdot \mu_2$ for $w_1, w_2 \in W$ and $\mu_1, \mu_2 \in D$.

Then $\mu_1 = -w_1^{-1} w_2 \cdot \mu_2$, so $\mu_1 \in D \cap (-\mathcal{U})$. So $\mu_1 = 0$, hence $\lambda = 0$. //

Theorem (D.) Let $\mathcal{G}$ be connected and unital ON-cyclic.

If $W = W(\mathcal{G})$ is infinite, then $\mathcal{U} \cap (-\mathcal{U}) = \{0\}$.

Reference: See [P12]

Proof: Suppose not. Then take $\mu \neq 0$ in $\mathcal{U} \cap (-\mathcal{U})$. So we can write $\mu = w_1 . \lambda' = -w_2 . \lambda$ for some $w_1, w_2 \in W$ and $\lambda, \lambda' \in D$.

Then $\lambda' = -w_1^{-1} w_2 . \lambda$, i.e. $\lambda' = -w . \lambda$.

Let $J = \{ i \in I_n \mid \lambda_i = 0 \}$. So, $i \in I_n \setminus J \Rightarrow \lambda_i > 0$, since $\lambda \in D$.

J is a proper subset of I_n since $\lambda \neq 0$.

Let $\beta \in \Phi^J := \{ \alpha \in \Phi^+ \mid \alpha \notin \operatorname{span}\{\alpha_j\}_{j \in J} \}$.

So, $\langle \lambda, \beta \rangle = \langle \lambda, \sum c_i \alpha_i \rangle$

$= \sum c_i \langle \lambda, \alpha_i \rangle > 0$.

Then, $\langle -w.\lambda, w.\beta \rangle = -\langle \lambda, \beta \rangle < 0$. But, $\lambda' = -w.\lambda \in D$,

so if $w.\beta \in \Phi^+$ we would have:

$$\langle -w.\lambda, w.\beta \rangle = \langle -w.\lambda, \sum c_i' \alpha_i \rangle = \sum \underbrace{c_i'}_{\geq 0} \underbrace{\langle -w.\lambda, \alpha_i \rangle}_{\geq 0} \geq 0$$

Thus, $w.\beta \in \Phi^-$.

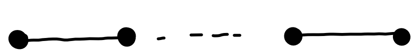
This shows that any $\beta \in \Phi^J$ is also in $N(w)$. Since $N(w)$ is finite, then Φ^J is finite. But this contradicts our finiteness / infiniteness theorem from p. 13 of Part 1 of Unit 3. //

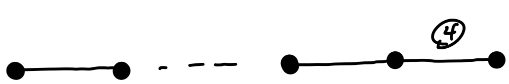
This is the reason for the unital ON-cyclic hypothesis.

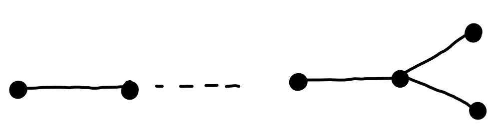
The above Theorem essentially rules out many possibilities for admissible graphs. An analysis of admissible graphs using the above Theorem and the Classification of finite Coxeter groups yields the following:

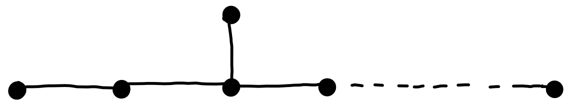
<u>Theorem (D.)</u> A connected SC-graph is admissible if and only if it is in one of the following mutually exclusive families of SC graphs:

Reference: See [P10]

A_n $(n \geq 1)$ 

B_n $(n \geq 2)$ 

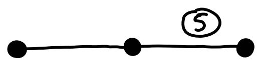
D_n $(n \geq 4)$ 

$\mathcal{E}_n$ $(n = 6, 7, 8)$ 

$\mathcal{F}_4$ 

$\mathcal{G}_2$ 

$\mathcal{H}_2$ 

$\mathcal{H}_3$ 

$\mathcal{H}_4$ 

$\mathcal{I}_2(m)$  , $m \geq 7$

NOTES: ① These are precisely the SC-graphs $\mathcal{G}$ for which the corresponding Coxeter groups $W = W(\mathcal{G})$ are finite.

② As a consequence, $\mathcal{U} = V^* = -\mathcal{U}$, so all numbers games played from any given start position converge to the same terminal position in the same number of moves.

Corollary (D.) $\mathcal{G}$ is connected and $W = W(\mathcal{G})$ is infinite

$$\Longleftrightarrow \mathcal{U} \cap (-\mathcal{U}) = \{0\}.$$