Coxeter groups and Combinatorics

Rob Donnelly
MSU
Oct/Nov 2009

Unit 2: More about Generators and Relations

PART 2

Throughout these notes, "References" come from the annoted references webpage linked to from the site where these notes are posted.

This unit will serve as a reminder/reintroduction to how algebraic objects can somtimes be usefully and succinctly described in terms of generators and relations, and how such descriptions can be very helpful in constructing morphisms between algebraic structures.

- Some linear algebra

Let V, W be real vector spaces (assume finite-dimensional)

A linear transformation from V to W is a "morphism" $T: V \to W$ such that:

① $T(u+v) = T(u) + T(v) \quad \forall\, u, v \in V$

② $T(cv) = c\,T(v) \quad \forall\, v \in V,\ c \in \mathbb{R}$

If $\{e_1, e_2, \ldots, e_n\}$ is a basis for V, then one can define a linear transformation $T: V \longrightarrow W$ by just declaring where the e_i's go. That is, any set mapping $\{e_1, e_2, \ldots, e_n\} \xrightarrow{f} W$ extends uniquely to a linear transformation $T: V \to W$, as pictured below:

<u>Picture</u>

$$\begin{tikzcd} \{e_1, \dots, e_n\} \arrow[r, hook, "i"] \arrow[dr, "f"'] & V \arrow[d, dashed, "T"] \\ & W \end{tikzcd}$$

$\exists!$ linear transformation $T: V \to W$ such that $f = T \circ i$

- <u>Invertible linear transformations</u>

i.e. There is an $S: V \to V$ such that $T \circ S = S \circ T = Id|_V$. Then write T^{-1} for S.

Let $GL(V) = \{$invertible linear transformations $T: V \to V\}$

$\boxed{\text{Exercise}}$ Prove that if $T: V \to V$ is an invertible linear transformation, then T is one-to-one and onto, and $T^{-1}: V \to V$ is one-to-one, onto, and linear.

<u>NOTE:</u> If $f: A \to B$ and $g: B \to A$ are set maps, one can show that if $f \circ g = id|_B$, then f is surjective and g is injective.

$\boxed{\text{Exercise}}$ Show that $GL(V)$ is a group under composition.

- <u>Representing linear transformations by matrices</u>

Let $\mathcal{B} = \{e_1, \dots, e_n\}$ be a basis for V.

For any $v \in V$, write $v = c_1 e_1 + c_2 e_2 + \dots + c_n e_n$, where each $c_i \in \mathbb{R}$.

Then let $[v]_{\mathcal{B}}$ be the column vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Let $T: V \to V$ be linear. Then let $[T]_{\mathcal{B}}$ be the matrix

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | & & | \\ [T(e_1)]_{\mathcal{B}} & [T(e_2)]_{\mathcal{B}} & \cdots & [T(e_n)]_{\mathcal{B}} \\ | & | & & | \end{bmatrix}$$

$\boxed{\text{Exercise}}$ Show that for any $v \in V$, $[T(v)]_{\mathcal{B}} = \underbrace{[T]_{\mathcal{B}} [v]_{\mathcal{B}}}_{\text{Matrix multiplication.}}$

FACT : T is one-to-one $\iff$ $[T]_{\mathcal{B}}$ is invertible $\iff$ T is onto.

$\boxed{\text{Exercise}}$ Let $v \in \mathbb{R}^2$, $v \neq 0$. So $L := \{tv \mid t \in \mathbb{R}\}$ is a line in $V = \mathbb{R}^2$.

Let $S: \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection of all elements of $\mathbb{R}^2$ across L. Convince yourself with a picture that S is linear.

Now find $[S]_{\mathcal{B}}$ for the basis $\mathcal{B} = \{(1,0), (0,1)\}$.

$\boxed{\text{Exercise}}$ Let $R_\theta: \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping that rotates the plane $V = \mathbb{R}^2$ through an angle θ. Convince yourself with a picture that R_θ is linear.

(counterclockwise if $\theta > 0$)

Now find $[R_\theta]_{\mathcal{B}}$ for the basis $\mathcal{B} = \{(1,0), (0,1)\}$.

$\boxed{\text{Exercise}}$ For $v_1, v_2 \in \mathbb{R}^2$, $v_1 \neq 0 \neq v_2$, let $L_i := \{t v_i \mid t \in \mathbb{R}\}$ for $i = 1, 2$.

Let S_i be the reflection across line L_i for $i = 1, 2$. Let θ be the counterclockwise angle from L_1 to L_2.

Show that $S_2 \circ S_1 = R_{2\theta}$.

- <u>The general linear group</u>

Let $GL(n,\mathbb{R})$ be the set of all invertible $n \times n$ matrices with real entries.

$\boxed{\text{Exercise}}$ Show that $GL(n,\mathbb{R})$ is a group under matrix multiplication.

$\boxed{\text{Exercise}}$ Show that $GL(V) \cong GL(n,\mathbb{R})$, an isomorphism of groups.

<u>Hint:</u> Fix a basis $\mathcal{B}$ of V. Then let $\varphi : GL(V) \to GL(n,\mathbb{R})$ be given by $\varphi(T) = [T]_{\mathcal{B}}$ ...

The group $GL(V)$ or $GL(n,\mathbb{R})$ is called the general linear group.

- <u>The usual inner product as a bilinear form</u>

→ The geometry of Euclidean space is encoded in the dot product on $\mathbb{R}^n$...

$v, w \in \mathbb{R}^n$ with $v = (v_1, \dots, v_n) = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $w = (w_1, \dots, w_n) = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$.

$$\Rightarrow \quad v \cdot w = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$$

→ Connection with geometry: $\|v\| = \sqrt{v \cdot v}$, $d(v,w) = \|v - w\|$,

and $\|v\| \|w\| \cos\theta = v \cdot w$

→ Bilinearity: $(cu+v)\cdot w = c(u \cdot w) + v \cdot w$, $v \cdot (cu + w) = c(v \cdot u) + v \cdot w$

→ Symmetric: $v \cdot w = w \cdot v$ for all $v, w \in \mathbb{R}^n$

→ Positive definite: $v \cdot v > 0$ for all $v \neq 0$ in $\mathbb{R}^n$

Exercise Prove that the dot product on $\mathbb{R}^n$ is symmetric and positive definite

→ <u>A matrix viewpoint</u>:

Let $E_{n \times n} = n \times n$ identity matrix. Let $\mathcal{B}$ = usual basis for $\mathbb{R}^n$.

Then $v \cdot w = [v]_{\mathcal{B}}^t \, E_{n \times n} [w]_{\mathcal{B}}$.

- <u>Bilinear forms in general</u>

→ A bilinear form on V is a function $B : V \times V \to \mathbb{R}$ for which

1. $B(cu + v, w) = cB(u,w) + B(v,w)$

2. $B(v, cu + w) = cB(v,u) + B(v,w)$

→ Fix a basis $\mathcal{B} = \{e_i\}_{i \in I_n}$ for V.

Let $M := \left(B(e_i, e_j) \right)_{i,j \in I_n}$ a square matrix with rows and columns indexed by I_n.

From here on, write $[B]_{\mathcal{B}}$ for M.

Then $B(v,w) = [v]_{\mathcal{B}}^t \, [B]_{\mathcal{B}} [w]_{\mathcal{B}}$

So, we only need to know what B does to basis vectors.

Exercise Prove the previous identity.

→ Definitions B is symmetric if $B(v,w) = B(w,v) \quad \forall\, v,w \in V$

B is positive definite if $B(v,v) > 0 \quad \forall\, v \in V$

B is positive semidefinite if $B(v,v) \geq 0 \quad \forall\, v \in V$

B is nondegenerate if for all $v \in V$, there is some $w \in V$ such that $B(v,w) \neq 0$.

FACT B is nondegenerate $\iff$ $[B]_{\mathcal{B}}$ is invertible.

→ Recall the theorem Dr. Ivansic told us about ("Sylvester's Theorem"):

If B is symmetric, there is a basis $\mathcal{B}$ for V with respect to which

$$[B]_{\mathcal{B}} = \left[\begin{array}{c|c|c} E_{k\times k} & 0 & 0 \\ \hline 0 & -E_{\ell\times\ell} & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

for some non-negative integers k and ℓ with $k+\ell \leq n$.

- Linear representations of groups

$\boxed{\text{Given a group } G, \text{ a linear representation of } G \text{ identifies each element of } G \text{ with an invertible linear transformation of a vector space } V, \text{ or equivalently with an invertible matrix.}}$

→ <u>Definition</u> A real, linear representation of a group G is a homomorphism $\rho : G \longrightarrow GL(V)$ for some real vector space V.

(If emphasizing a matrix viewpoint, then consider $\rho : G \longrightarrow GL(n, \mathbb{R})$.)

→ <u>Example</u> Let $G = \langle A, t \mid A^2 = t^2 = (At)^4 = \varepsilon \rangle$, dihedral of order 8.

Let $\rho : G \longrightarrow GL(\mathbb{R}^2)$ be given by

$\rho(A)$ = "reflection across the line $y = x$" $=: S$

$\rho(t)$ = "reflection across x-axis" $=: T$

$\boxed{\text{Exercise}}$ Check that $S^2 = T^2 = (ST)^4 = Id|_{\mathbb{R}^2}$

→ <u>Notation, G-modules</u>

For each $g \in G$, $\rho(g) : V \to V$ is an invertible linear transformation.

<u>Notation:</u> $\rho(g)(v)$ is also written $g.v$ (for each $v \in V$)

<u>Property 1:</u> $\rho(\varepsilon) = Id$, so $\varepsilon . v = v$ for all $v \in V$.

<u>Property 2:</u> $\rho(gh)(v) = (\rho(g) \circ \rho(h))(v)$, so

$$(gh).v = g.h.v \qquad \text{for all } v \in V, \ g,h \in G.$$

If you are working mostly with the "lower-dot" notation, you'd refer to V as a "G-module" and talk about "G acting on V."

- <u>The "dual" space V^*</u>

→ Given a real vector space V.

Let $V^* = \{ \text{linear transformations } f : V \to \mathbb{R} \}$ "linear functionals"

Then V^* is a vector space with ...

... addition $f+g$, usual addition of real-valued functions

... scalar multiplication cf, usual multiplication of a real-valued function by a real scalar c.

→ <u>Example</u> If $V = \mathbb{R}$, then $V^* = \{ y = mx \mid m \in \mathbb{R} \}$.

<u>Reason:</u> If $f : \mathbb{R} \to \mathbb{R}$ is linear, then let $m = f(1)$.

Then $y = f(x) = x\, f(1) = x \cdot m = mx$.

→ <u>Proposition</u> If $\mathcal{B} = \{e_1, \ldots, e_n\}$ is a basis for V, then $\mathcal{B}^* := \{f_1, \ldots, f_n\}$ is a basis for V^*, where f_i is the linear functional $f : V \to \mathbb{R}$ determined by $f_i(e_j) = \delta_{ij}$.

$\mathcal{B}^*$ is the "dual" basis

$\boxed{\text{Exercise}}$ Prove this proposition.

<u>Hint:</u> For $f \in V^*$, let $c_i = f(e_i)$. Then show that $f = c_1 f_1 + \cdots + c_n f_n$, etc.

→ <u>Example</u> Let $V = \mathbb{R}^3$ and let $\mathcal{B} = \{\underset{e_1}{\vec{i}}, \underset{e_2}{\vec{j}}, \underset{e_3}{\vec{k}}\}$

Let $\mathcal{B}^* = \{f_1, f_2, f_3\}$ be the dual basis.

Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$.

Then
$$f_1(v_1, v_2, v_3) = v_1$$
$$f_2(v_1, v_2, v_3) = v_2$$
$$f_3(v_1, v_2, v_3) = v_3$$

→ There is a natural way to calculate with elements of V^* and V:

For $f \in V^*$ and $v \in V$, $\langle f, v \rangle := f(v)$.

- <u>The "dual" or "contragredient" representation</u>

→ Given $\rho : G \longrightarrow GL(V)$, define $\rho^* : G \longrightarrow GL(V^*)$ by ...

$\rho^*(g)$ is the linear transformation $V^* \longrightarrow V^*$ for which

$\rho^*(g)(f)$ is the linear functional whereby

$$\rho^*(g)(f)(v) := f(g^{-1}.v).$$

$\boxed{\text{Exercise}}$ ① Show that $\rho^*(gh) = \rho^*(g)\rho^*(h)$

② Show that $\rho^*(g) \in GL(V^*)$

③ For all $f \in V^*$, $v \in V$, and $g \in G$, show that

$$\underbrace{\langle g.f, v \rangle}_{\substack{G\text{-action on } V^* \\ \text{via } \rho^*}} = \underbrace{\langle f, g^{-1}.v \rangle}_{\substack{G\text{-action on } V \\ \text{via } \rho}}$$

④ ρ is injective $\iff$ ρ^* is injective.

→ It is a fact that if $\mathcal{B}$ is a basis for V and $\mathcal{B}^*$ is the dual basis for V^*, then $[\rho^*(g)]_{\mathcal{B}^*} = \left([\rho(g)]_{\mathcal{B}}^{-1}\right)^t$.

$\boxed{\text{Exercise}}$ See if you can show this.