Coxeter groups and Combinatorics

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Unit 2: More about Generators and Relations — PART 1

Throughout these notes, "References" come from the annoted references webpage linked to from the site where these notes are posted.

This unit will serve as a reminder/reintroduction to how algebraic objects can somtimes be usefully and succinctly described in terms of generators and relations, and how such descriptions can be very helpful in constructing morphisms between algebraic structures.

- A "universal property" for free groups

Let $\mathcal{S}$ be a set.

Let $\mathcal{F}_{\mathcal{S}}$ denote the free group on this set.

Note that $\mathcal{S}$ is a subset of $\mathcal{F}_{\mathcal{S}}$, which we depict as: $\mathcal{S} \overset{i}{\hookrightarrow} \mathcal{F}_{\mathcal{S}}$.

It can be seen that $\mathcal{S} \overset{i}{\hookrightarrow} \mathcal{F}_{\mathcal{S}}$ has the following "universal property":

If $\mathcal{S} \xrightarrow{f} G$ is any function mapping $\mathcal{S}$ to a group G,

then $\exists!$ homomomorphism $\varphi : \mathcal{F}_{\mathcal{S}} \longrightarrow G$ such that $f = \varphi \circ i$.

Picture

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathcal{F}_{\mathcal{S}} \arrow[d, dashed, "\varphi"] \\ & G \end{tikzcd}$$

$\boxed{\text{Exercise}}$

Prove the existence of the unique homomorphism φ.

HINT: If $s_{i_1} \cdots s_{i_k}$ is a word in the generating set $\mathcal{S} \cup \mathcal{S}^{-1}$, what must $\varphi(s_{i_1} \cdots s_{i_k})$ be?

- Some remarks about this universal property:

  * This means that any set mapping $\mathcal{S} \xrightarrow{f} G$ from $\mathcal{S}$ to a group G "extends uniquely" to a homomorphism $\mathcal{F}_{\mathcal{S}} \xrightarrow{\varphi} G$.

  * The group $\mathcal{F}_{\mathcal{S}}$ and the mapping $\mathcal{S} \xhookrightarrow{i} \mathcal{F}_{\mathcal{S}}$ are unique in the following sense:

    Suppose $\mathcal{S} \xrightarrow{j} H$ is a set mapping from $\mathcal{S}$ to a group H that enjoys this same universal property:

$$\begin{tikzcd} \mathcal{S} \arrow[r, "j"] \arrow[dr, "f"'] & H \arrow[d, dashed, "\psi"] \\ & G \end{tikzcd}$$

    Then j is injective and $H \cong \mathcal{F}_{\mathcal{S}}$.

$\boxed{\text{Exercise}}$ Prove the previous claim.

<u>Hint</u>: In case of

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathcal{F}_{\mathcal{S}} \arrow[d, dashed, "\varphi"] \\ & G \end{tikzcd}$$

, let $G = H$ and $f = j$.

Then for
$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "f"'] & H \arrow[d, dashed, "\psi"] \\ & G \end{tikzcd}$$
, let $G = \mathcal{F}_S$ and $f = i$.

Why is it the case now that $\psi \circ \varphi = id|_{\mathcal{F}_S}$?

Similarly argue that $\varphi \circ \psi = id|_H$.

- <u>Group presentations</u>

S = set , R = set of relations

Let $G = \langle S \mid R \rangle$

> <u>Example</u> $S = \{s, t\}$, $R = \{s^2, t^2, (st)^4\}$
>
> Then $G = \langle S \mid R \rangle$ is the 8-element dihedral group

We have the natural mapping $S \xrightarrow{r} G$ for which $r(s) = s$ for each $s \in S$. Then :

$$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[dr, "r"'] & \mathcal{F}_S \arrow[d, dashed, "\pi"] \\ & G \end{tikzcd}$$

<u>i.e. $r = \pi \circ i$</u>

<u>Fact:</u> π is surjective and $\mathcal{R} \subseteq \ker \pi$.

<u>Fact:</u> The mapping $\mathcal{F}_{\mathcal{S}} \xrightarrow{\pi} G$ is "universal" in the following sense: If $\mathcal{S} \xrightarrow{f} H$ is any mapping for which the induced homomorphism $\mathcal{F}_{\mathcal{S}} \xrightarrow{\varphi} H$ has $\mathcal{R} \subseteq \ker \varphi$, then $\exists!\ \psi : G \to H$ such that $\psi \circ \pi \circ i = f$, in which case $\psi \circ \pi = \varphi$.

<u>FACT:</u> $G \cong \mathcal{F}_{\mathcal{S}} / \langle \mathcal{R} \rangle$, where $\langle \mathcal{R} \rangle$ is the intersection of all normal subgroups of $\mathcal{F}_{\mathcal{S}}$ containing $\mathcal{R}$.

<u>Picture</u>

$$\begin{tikzcd}
\mathcal{S} \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathcal{F}_{\mathcal{S}} \arrow[r, two heads, "\pi"] \arrow[d, dashed, "\varphi"] & G \arrow[dl, dashed, "\psi"] \\
& H &
\end{tikzcd}$$

$\boxed{\text{Thought Question}}$ In what sense is $\mathcal{F}_{\mathcal{S}} \xrightarrow{\pi} G$ unique?

- <u>An interpretation</u>

We're saying here that whenever $\mathcal{S}$ is mapped to a group H in such a way that all the relations in $\mathcal{R}$ are "preserved" (i.e. $\mathcal{R} \subseteq \ker \varphi$), then this set mapping extends uniquely to a homomorphism from G to H, i.e.

<u>Principle</u>: Say where the generators go, check that relations are preserved, get a homomorphism

- *Example* $G = \langle \underbrace{s, t}_{S} \mid \underbrace{s^2 = t^2 = (st)^4}_{R} = \varepsilon \rangle$

$H = \{\pm 1\}$, with multiplication as the group operation for H.

Define $f : S \to H$ by the rule $f(s) = f(t) = -1$.

$\boxed{\text{Want: A homomorphism } \psi : G \to H \text{ for which } \psi(s) = \psi(t) = -1.}$

So, look at this picture

$$\begin{tikzcd} S = \{s, t\} \arrow[r, hook, "i"] \arrow[dr, "f"'] & F_S \arrow[d, dashed, "\varphi"] \\ & H = \{\pm 1\} \end{tikzcd}$$

Q: Is $R \leq \ker \varphi$?

A: Yes ... $\varphi(s^2) = \varphi(s)\,\varphi(s)$
$= f(s)\,f(s)$
$= (-1)\,(-1) = 1$

and similarly for $\varphi(t^2)$
and $\varphi[(st)^4]$.

So $\exists!$ homomorphism $\psi : G \to H$ for which $\psi(s) = -1$ and $\psi(t) = -1$:

$$\begin{tikzcd} S = \{s, t\} \arrow[r, hook, "i"] \arrow[drr, bend right] & F_S \arrow[r, two heads, "\pi"] \arrow[dr, dashed, "\varphi"] & G \arrow[d, dashed, "\psi"] \\ & & H = \{\pm 1\} \end{tikzcd}$$

So, there is a unique homomorphism from G to H which sends s and t to -1.

→ Nowhere in this set-up did we need to know that G is finite.

→ But ... What does ψ do to the other elements of G?

- Example $\quad W = \langle S \mid R \rangle \quad$ Coxeter

$S = \{s_i\}_{i \in I_n}$, $\quad I_n$ is an index set, $|I_n| = n$.

$$R = \left\{ (s_i s_j)^{m_{ij}} \;\middle|\; \begin{array}{c} m_{ii} = 1, \quad m_{ij} \in \{\infty, 2, 3, 4, \dots\} \\ \text{for } i \neq j, \text{ and } m_{ij} = m_{ji} \end{array} \right\}$$

Consider: $S \xrightarrow{f} \{\pm 1\}$ given by $s_i \xmapsto{f} -1 \;\; \forall i \in I_n$.

$\boxed{\text{Exercise}}$ For the induced map $\varphi : F_S \to \{\pm 1\}$, $R \subseteq \ker \varphi$.

(Thus there is a unique homomorphism $W \xrightarrow{sgn} \{\pm 1\}$ which sends all generators to -1, $s_i \xmapsto{sgn} -1$.)

Corollary: For each $i \in I_n$, $s_i \neq \varepsilon$ and s_i has order 2 in W.

Proof: Since $sgn(s_i) = -1 \neq 1$, then we must have $s_i \neq \varepsilon$.

Since $s_i^2 = \varepsilon$ in W and $s_i \neq \varepsilon$, then s_i has order 2. //

$\boxed{\text{Thought Questions}}$ ① Might $s_i = s_j$ in W for some $i \neq j$?

NOTE: In $G = \langle s, t \mid s^2 = t^2 = (st)^3 = (ts)^2 = \varepsilon \rangle$, it is the case that $s = t$. (Why?)

② Does $s_i s_j$ have order m_{ij} in W for $i \neq j$?

NOTE: For the above G, $st = \varepsilon$ has order 1.

- On the Word Problem for Coxeter groups

$W = \langle S \mid R \rangle$ Coxeter, as above, with $S = \{s_i\}_{i \in I_n}$

Let $I_n^* =$ set of all finite sequences from I_n, including the empty sequence $(\ )$.

For $x \in I_n^*$ with $x = (j_1, \dots, j_k)$, say $\text{length}(x) = k$.

Two elementary simplifications:

$\boxed{\text{Length-reducing}}$ Replace subword (i, i) with empty

$\boxed{\text{Braid}}$ Replace $\underbrace{(i, j, i, j, \dots)}_{\text{Length } m_{ij}}$ with $\underbrace{(j, i, j, i, \dots)}_{\text{Length } m_{ij}}$

Ex: If $m_{ij} = 3$, then (i, j, i) is replaced by (j, i, j).

Let $x \in I_n^*$, and let $S(x) := \left\{ \begin{array}{l} \text{sequences obtainable from } x \text{ by some} \\ \text{(possibly empty) sequence of elementary} \\ \text{simplifications} \end{array} \right\}$

Ex: $S(i, i, j, i) = \{ (i, i, j, i), (j, i), (i, j, i, j), (j, i, j, j) \}$

if $m_{ij} = 3$

Observation: If $y \in S(x)$, then $\text{length}(y) \leq \text{length}(x)$.

Let $T : I_n^* \longrightarrow W$ be given by $T(j_1, \ldots, j_k) := s_{j_k} \cdots s_{j_1}$

<u>Observation</u>: If $y \in \mathcal{S}(x)$, then $T(y) = T(x)$.

<u>Tits' Theorem for the Word Problem on Coxeter Groups</u>

Reference: See [P7]

For $x, y \in I_n^*$, $\quad T(x) = T(y) \iff \mathcal{S}(x) \cap \mathcal{S}(y) \neq \phi$.

<u>NOTES</u>: ① "$\Leftarrow$" is easy, "$\Rightarrow$" is hard.

② This answers Thought Question ① above --- how?

<u>Corollary 1</u>: Let $x \in I_n^*$ and let $y = (i_1, \ldots, i_p)$ be a shortest length sequence in $\mathcal{S}(x)$. Let $w = T(x)$. Then, $w = s_{i_p} \cdots s_{i_1}$, and any expression of w as a product of generators must use at least p generators.

<u>Interpretation</u>: If this $w \in W$ is expressed as $w = s_{j_k} \cdots s_{j_1}$, then a "shortest" expression $s_{i_p} \cdots s_{i_1}$ $(p \leq k)$ can be obtained by applying some sequence of elementary simplifications.

$\boxed{\text{Exercise}}$ Prove Corollary 1.

<u>Corollary 2</u>: Let $x = (j_1, \ldots, j_k) \in I_n^*$, and let $w = T(x)$.

Then $w = \varepsilon \iff$ empty sequence $\in \mathcal{S}(x)$.

$\boxed{\text{Exercise}}$ Prove Corollary 2.

- The length function

Take $W = \langle \mathcal{S} \mid \mathcal{R} \rangle$ Coxeter, as above.

For $w \in W$, the length of w is the smallest number p such that w can be written as a product of p generators.

If $\ell(w) = p$ and $w = s_{i_p} \cdots s_{i_1}$, then we say $s_{i_p} \cdots s_{i_1}$ is reduced.

Facts: (L-1) $\ell(w) = p \iff$ whenever $T(i_1, \ldots, i_p) = w$, then $(i_1, \ldots, i_p)$ is shortest in any $S(x)$ for which $T(x) = w$.

(L0) $\ell(w) = 0 \iff w = \varepsilon$ (by definition an empty product is ε)

(L1) $\ell(w) = \ell(w^{-1})$

(L2) $\ell(w) = 1 \iff w = s_i$ for some $i \in I_n$

(L3) $\ell(ww') \leq \ell(w) + \ell(w')$

NOTE: $0 = \ell(s_i s_i) < \ell(s_i) + \ell(s_i) = 2$, so the inequality can be strict.

(L4) $\ell(ww') \geq \ell(w) - \ell(w')$

NOTE: For $i \neq j$, $2 = \ell(s_i s_j) > \ell(s_i) - \ell(s_j) = 1 - 1 = 0$, so the inequality can be strict.

(L5) $\ell(w) - 1 \leq \ell(ws_i) \leq \ell(w) + 1$

$\ell(w) - 1 \leq \ell(s_i w) \leq \ell(w) + 1$

Exercises: Prove (L1), (L2), and (L3).

Proposition: Suppose $w = s_{j_k} \cdots s_{j_1}$. Then $\ell(w)$ and k have the same parity.

$\boxed{\text{Exercise}}$ Prove this using Corollary 1 of Tits' Theorem.

Now recall the homomorphism $\mathrm{sgn}: W \to \{\pm 1\}$ we found for which $\mathrm{sgn}(s_i) = -1$ for each $i \in I_n$.

Proposition: For any $w \in W$, $\mathrm{sgn}(w) = (-1)^{\ell(w)}$.

Proof: Let $\Psi : W \to \{1\}$ be given by $\Psi(w) := (-1)^{\ell(w)}$.

Say $w_1 = s_{i_p} \cdots s_{i_1}$ (reduced)

and $w_2 = s_{j_q} \cdots s_{j_1}$ (reduced).

Then $w_1 w_2 = (s_{i_p} \cdots s_{i_1})(s_{j_q} \cdots s_{j_1})$, so by the above Proposition, $\ell(w_1 w_2)$ and $p+q$ have the same parity.

Then $\Psi(w_1 w_2) = (-1)^{\ell(w_1 w_2)} = (-1)^{p+q} = (-1)^p (-1)^q = \Psi(w_1)\Psi(w_2)$.

So, Ψ is a homomorphism. Since sgn was obtained as the unique homomorphism $W \to \{\pm 1\}$ for which each $s_i \mapsto -1$, we conclude that $\Psi = \mathrm{sgn}$. //

Corollary: $\ell(ws_i) = \ell(w) \pm 1$, $\ell(s_i w) = \ell(w) \pm 1$

Proof: $(-1)^{\ell(ws_i)} = \psi(ws_i) = \psi(w)\,\psi(s_i) = (-1)^{\ell(w)} \cdot (-1) = (-1)^{\ell(w)+1}$.

In particular $\ell(ws_i) \neq \ell(w)$.

Since $\ell(w) - 1 \leq \ell(ws_i) \leq \ell(w) + 1$ by (L5),

and since $\ell(ws_i) \neq \ell(w)$, then $\ell(ws_i) = \ell(w) \pm 1$.

Similarly, $\ell(s_i w) = \ell(w) \pm 1$. //

- <u>The symmetric groups from abstract algebra</u>

The symmetric group S_n from abstract algebra is a Coxeter group!

$$S_n \cong \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{l} (s_i)^2 = \varepsilon, \text{ and for } i \neq j \text{ we have} \\ (s_i s_j)^3 = \varepsilon \text{ if } i = j \pm 1 \text{ and} \\ s_i s_j = s_j s_i \text{ otherwise} \end{array} \right\rangle =: W$$

with isomorphism $f : W \to S_n$ determined by $f(s_i) = (i, i+1)$

adjacent transposition

$\boxed{\text{Exercise}}$ Show this.

$\boxed{\text{Exercise}}$ What is ker(sgn) for the dihedral groups?

The homomorphism $\mathrm{sgn}(w) = (-1)^{\ell(w)}$ found above coincides with the "sign representation" on S_n: $\mathrm{sgn}(\sigma) = \begin{cases} -1 & \text{if } \sigma \text{ is even} \\ 1 & \text{if } \sigma \text{ is odd} \end{cases}$

The alternating normal subgroup $A_n \trianglelefteq S_n$ is just ker(sgn).

So every Coxeter group has a version of the alternating subgroup ... ker(sgn).