Coxeter groups and Combinatorics

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Unit 1 : The Numbers Game

Throughout these notes, "References" come from the annoted references webpage linked to from the site where these notes are posted.

The numbers game is a one-player game played on any finite simple graph whose edges are allowed to be weighted in certain ways. This game has been independently invented several times, but we will consider the version studied by Kimmo Eriksson. We will see in Unit 3 of these talks that the game is a model for certain geometric representations of Coxeter groups. (These groups are named after H.S.M. Coxeter, a Canadian and great 20th century geometer who is famous for his work with regular polytopes.) This interplay between combinatorics and algebra will help us answer, by way of a classification result, a finiteness question about the numbers game. The answer to this finiteness question has also helped answer related questions about finite posets and distributive lattices that arise in the study of Weyl characters, cf. Unit 4.

- Let Γ be a finite simple graph (no loops, no multiple edges) with n nodes indexed by a set I_n:

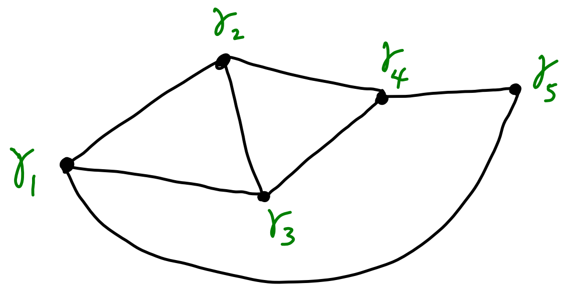
  $\mathcal{V}(\Gamma)$ = vertex set of Γ = $\{\gamma_i\}_{i \in I_n}$

  $\mathcal{E}(\Gamma)$ = edge set of Γ

  * "$\{\gamma_i, \gamma_i\}$" $\notin \mathcal{E}(\Gamma)$ since Γ has no loops

  * $\{\gamma_i, \gamma_j\}$ $(i \neq j)$ appears in $\mathcal{E}(\Gamma)$ at most once since Γ has no multiple edges.

- <u>Example</u>

$\Gamma =$ 

$\{\gamma_2, \gamma_4\} \in \mathcal{E}(\Gamma)$

$\{\gamma_1, \gamma_5\} \in \mathcal{E}(\Gamma)$

etc.

- For convenience in future calculations ...

$$A := (a_{ij})_{i,j \in I_n} \qquad a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } \{\gamma_i, \gamma_j\} \in \mathcal{E}(\Gamma) \\ 0 & \text{otherwise} \end{cases}$$

Call A the "amplitude" matrix. (It's a sort-of incidence or adjacency matrix.)

From here on, the pair $\mathcal{G} := (\Gamma, A)$ will be our "game graph"

- A position λ is an assignment of real numbers to the nodes of $\mathcal{G}$:

$$\lambda = (\lambda_i)_{i \in I_n}, \qquad \lambda_i \in \mathbb{R}$$

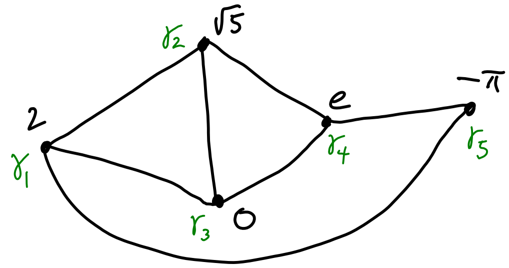
λ is dominant if each $\lambda_i \geq 0$.

λ is strongly dominant if each $\lambda_i > 0$.

λ is trivial if each $\lambda_i = 0$.

λ is nontrivial if at least one $\lambda_i \neq 0$.

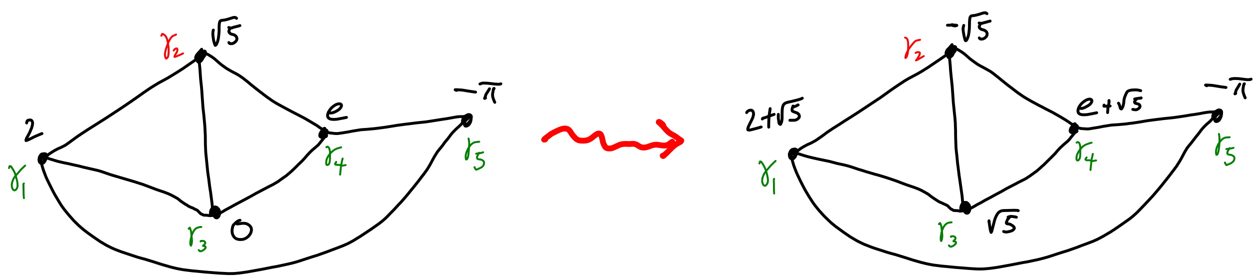
- Example



- Given a position λ on $\mathcal{G}$, to "fire" node γ_i is to change the number at each node by the transformation

$$\lambda_j \longmapsto \lambda_j - a_{ij}\lambda_i \qquad \forall\, j \in I_n$$

Example Fire node γ_2 ---



$\lambda = (2, \sqrt{5}, 0, e, -\pi)$ $\qquad$ $\lambda^{NEW} = (2+\sqrt{5}, -\sqrt{5}, \sqrt{5}, e+\sqrt{5}, -\pi)$

- The numbers game is the one-player game on $\mathcal{G} = (\Gamma, A)$ in which the player:

(0) Assigns an initial position λ to $\mathcal{G}$;

(1) Chooses a node γ_i with positive number $\lambda_i > 0$ and fires;

and (2) From this new position λ repeats step (1) if the new position has a positive number at some node.

- <u>Some terminology</u>

→ The complete (and possibly infinite) sequence $(\gamma_{i_1}, \gamma_{i_2}, \gamma_{i_3}, \ldots)$ of fired nodes is the game sequence.

→ A game sequence is convergent if it is finite, in other words, the numbers game terminated at some position where all numbers on all nodes were nonpositive. Otherwise, divergent.

→ In general we'll use the word "legal" with respect to a sequence of node firings to emphasize that each firing was allowed.

[That is, a sequence of node firings is legal if it is a subsequence of consecutive nodes fired in some game sequence.]

- <u>Finiteness Question</u>

For which connected game graphs $\mathcal{G} = (\Gamma, A)$ and nontrivial dominant initial positions λ is there a convergent (i.e. terminating) game sequence?

- **Example**

	γ_1	γ_2
	2	5
γ_1	-2	7
γ_2	5	-7
γ_1	-5	-2

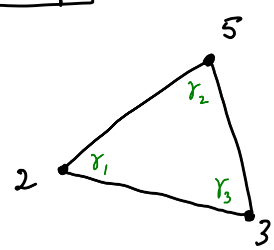
Same terminal position

	γ_1	γ_2
	2	5
γ_2	7	-5
γ_1	-7	2
γ_2	-5	-2

Game seq $= (\gamma_1, \gamma_2, \gamma_1)$

Game seq $= (\gamma_2, \gamma_1, \gamma_2)$

- **Example**



γ_1: -2, 7, 5

γ_2: 5, -7, 12

γ_1: -5, -2, 17

γ_3: 12, 15, -17

$(\gamma_1, \gamma_2, \gamma_1)$: -15, -12, 37

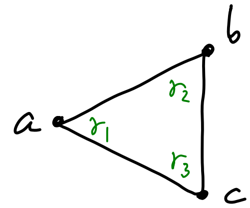
γ_3: 22, 25, -37

$\vdots$

It appears this game sequence is not convergent...

but maybe a different firing sequence would converge??

- $\boxed{\text{Exercise}}$ From initial position



$(a>0,\ b>0,\ c>0)$,

Show that the firing sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_3)$ can be legally repeated indefinitely.

[That is, show that $(\gamma_1, \gamma_2, \gamma_1, \gamma_3, \gamma_1, \gamma_2, \gamma_1, \gamma_3, \ldots)$ is a divergent game sequence.]

- A generalization ...

Given our finite simple graph Γ as before, let $A := (a_{ij})_{i,j \in I_n}$

where
$$a_{ij} = \begin{cases} 2 & \text{if } i=j \\ \boxed{\text{Some real } \#} & \text{if } \{\gamma_i, \gamma_j\} \in \mathcal{E}(\Gamma) \\ 0 & \text{otherwise} \end{cases}$$

and at least one of a_{ij} or a_{ji} is nonzero if $\{\gamma_i, \gamma_j\} \in \mathcal{E}(\Gamma)$.

Play the numbers game as before, where firing node γ_i from position λ is still the transformation $\lambda_j \longmapsto \lambda_j - a_{ij}\lambda_i$, $\forall j \in I_n$.

- A limitation ...

  For our game graph $\mathcal{G} = (\Gamma, A)$ and each initial position λ,

  we want all game sequences either to

  | ① All diverge, or
  |
  | ② All converge to the same terminal position
  | in the same # of steps.

  Eriksson calls this property (of game graphs) strong convergence.

  Call a game graph an SC-graph if it is strongly convergent.

- Reasons for this limitation ...

  Makes our finiteness question a bit more tractable ...

  Allows us to bring Coxeter group theory into the picture ...

- Finiteness Question (revised)

  For which connected SC-graphs $\mathcal{G} = (\Gamma, A)$

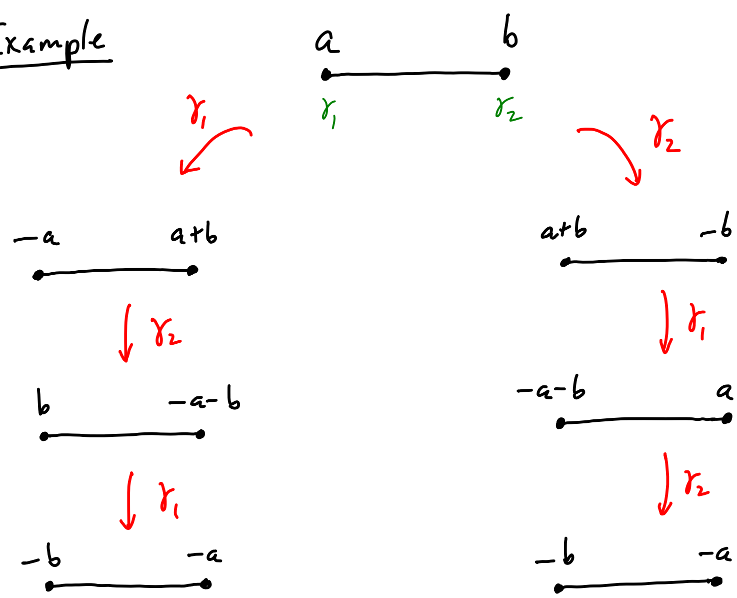
  and nontrivial dominant initial positions λ

  is there a convergent (i.e. terminating) game sequence?

- <u>Another Question</u>

  How to characterize the set of positions for an SC-graph for which there is a convergent game sequence?

- <u>Example</u>



$$\mathcal{G} = \left(\underset{\gamma_1}{\bullet}\!\!-\!\!-\!\!-\!\!\underset{\gamma_2}{\bullet},\ \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \right)$$

This game graph is of "type A_2"

- $\boxed{\text{Exercise}}$ Use the above calculations to prove that the above is an SC-graph.

- <u>Notation</u> From here on, depict $\left(\underset{\gamma_1}{\bullet}\!\!-\!\!-\!\!-\!\!\underset{\gamma_2}{\bullet},\ \begin{bmatrix} 2 & -p \\ -q & 2 \end{bmatrix} \right)$ as $\underset{\gamma_1}{\bullet} \xrightarrow[p]{} \xleftarrow[q]{} \underset{\gamma_2}{\bullet}$

- $\boxed{\text{Exercise}}$ Using reasoning similar to the previous exercise, show that $\underset{\gamma_1}{\bullet} \xrightarrow[1]{} \xleftarrow[3]{} \underset{\gamma_2}{\bullet}$ is an SC-graph.

- <u>Theorem (Eriksson, 1996)</u> $\mathcal{G} = (\Gamma, A)$ is an SC-graph $\iff$

  ① For all $i, j \in I_n$, if $i \neq j$ then $a_{ij} \leq 0$

  ② For all $i, j \in I_n$, if $i \neq j$ then

  $$a_{ij} \neq 0 \iff a_{ji} \neq 0$$

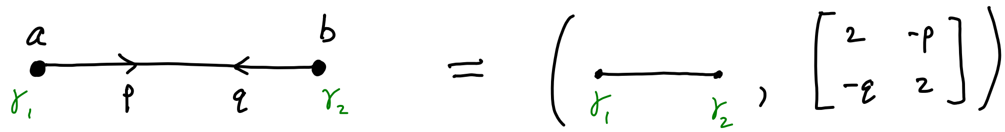
  ③ For all $i, j \in I_n$, if $i \neq j$ then

  either $a_{ij}\, a_{ji} = 4\cos^2(\pi / m_{ij})$ for some integer $m_{ij} \geq 2$

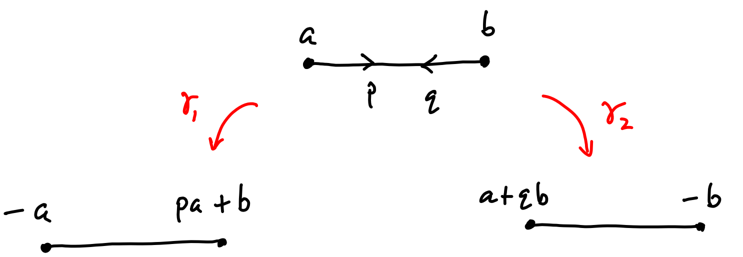
  or $a_{ij}\, a_{ji} \geq 4$ (in which case we can take $m_{ij} = \infty$)

Reference: See [P15], [P17]

- First, an analysis of two-node games ...



$\boxed{\text{Case 1}}$ <u>$p < 0$ and $q < 0$</u>



We can choose a and b positive such that $pa + b > 0$ but $a + qb \leq 0$
(e.g. take $-\frac{b}{a} \ll 0$ so $-\frac{b}{a} < p$, and take $-\frac{a}{b}$ near 0 so $q \leq -\frac{a}{b}$).
So this is not an SC graph.

$\boxed{\text{Case 2}}$ $\underline{p < 0 \quad \text{and} \quad q \geq 0}$

$\boxed{\text{Case 3}}$ $\underline{p > 0 \quad \text{and} \quad q = 0}$

$\boxed{\text{Exercise}}$ In Cases 2 and 3 above, argue that the game graph $\bullet \xrightarrow[p]{} \xleftarrow[q]{} \bullet$ is NOT an SC-graph.

- <u>A digression on "alternating Fibonacci polynomials"</u>

Consider the family of polynomials $\{r_k(x)\}_{k=0}^{\infty}$ determined by the recurrence: $r_0(x) = r_1(x) = 1$, and for $k \geq 2$,

$$r_k(x) = \begin{cases} r_{k-1}(x) - r_{k-2}(x) & \text{if } k \text{ is odd} \\ x\, r_{k-1}(x) - r_{k-2}(x) & \text{if } k \text{ is even} \end{cases}$$

For convenience later, take $r_{-1}(x) := 0$.

A related family of polynomials $\{P_k(x)\}_{k=0}^{\infty}$ is defined by

$$P_k(x) = \sum_{j \geq 0} \binom{k-j}{j} x^j$$

Here's a picture of $P_k(x)$ using Pascal's triangle:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

$$P_7(x) = \binom{7}{0}\cdot 1 + \binom{6}{1}x + \binom{5}{2}x^2 + \binom{4}{3}x^3$$
$$= 1 + 6x + 10x^2 + 4x^3$$

Claim: $r_k(x) = x^{\lfloor \frac{k}{2} \rfloor} P_k\left(-\frac{1}{x}\right)$.

If so, then $r_7(x) = x^3\left(1 - \frac{6}{x} + \frac{10}{x^2} - \frac{4}{x^3}\right)$
$= x^3 - 6x^2 + 10x - 4$

In general, $r_k(x)$ is a monic polynomial.

Exercise Prove this claim by showing that $Q_k(x) := x^{\lfloor k/2 \rfloor} P_k\left(-\frac{1}{x}\right)$ satisfies the same recurrence as $r_k(x)$.

Also show that $P_k(x) = P_{k-1}(x) + x P_{k-2}(x)$ for all $k \geq 2$, with $P_0(x) = P_1(x) = 1$.

Exercise Show that $\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$

Proposition (Benoumhani, 2003)

Reference: See [P2]

$$P_{k-1}(x) = 0 \iff x = -\frac{1}{4}\left(1 + \tan^2\left(\frac{j\pi}{k}\right)\right) \quad \text{for} \quad 1 \leq j \leq \left\lfloor \frac{k-1}{2} \right\rfloor.$$

Then $r_{k-1}(x) = 0 \iff x = 4\cos^2\left(\frac{j\pi}{k}\right)$ for $1 \leq j \leq \left\lfloor \frac{k-1}{2} \right\rfloor$,

in which case $0 < x < 4$.

- <u>Proposition</u> Consider the two-node game graph $\underset{\gamma_1}{\bullet} \xrightarrow{\ p\ } \xleftarrow{\ q\ } \underset{\gamma_2}{\bullet}$ with $p, q > 0$.

  Take a strongly dominant initial position (a, b), so $a, b > 0$.

  If $pq = 4\cos^2(\pi/m)$ for some integer $m \geq 3$, then we have strong convergence in m node firings.

  If $pq \geq 4$, then all game sequences are divergent.

<u>Outline of proof:</u>

For k firings $(k \geq 1)$ from $\overset{a}{\bullet} \xrightarrow{\ p\ } \xleftarrow{\ q\ } \overset{b}{\bullet}$ beginning with γ_1,

the resulting position is $\overset{u(k)}{\bullet} \xrightarrow{\ p\ } \xleftarrow{\ q\ } \overset{v(k)}{\bullet}$ with

$$u(k) = \begin{cases} r_k(pq)\,a + q\,r_{k-1}(pq)\,b & \text{if } k \text{ is even} \\ -r_{k-1}(pq)\,a - q\,r_{k-2}(pq)\,b & \text{if } k \text{ is odd} \end{cases}$$

$$v(k) = \begin{cases} -p\,r_{k-1}(pq)\,a - r_{k-2}(pq)\,b & \text{if } k \text{ is even} \\ p\,r_k(pq)\,a + r_{k-1}(pq)\,b & \text{if } k \text{ is odd} \end{cases}$$

This can be easily shown by induction.

So now, if $pq \geq 4$, then one can see by inspection that at least one of $u(k)$ or $v(k)$ is positive. Hence the game sequence $(\gamma_1, \gamma_2, \gamma_1, \ldots)$ is divergent.

Now suppose $pq = 4\cos^2(\pi/m)$ with $\underline{\underline{m \text{ even}}}$. Then notice that $4\cos^2(\pi/j) < 4\cos^2(\pi/m)$ for all $j < m$.

In particular, $r_{j-1}(pq) > 0$ for all $j < m$, so the firing sequence $\underbrace{(\gamma_1, \gamma_2, \gamma_1, \ldots, \gamma_2)}_{\text{Length} = m}$ is legal.

This brings us to position:

$$u(m) = r_m(pq)\,a + q\,r_{m-1}(pq)\,b$$

$$v(m) = -p\,r_{m-1}(pq)\,a - r_{m-2}(pq)\,b$$

Now $r_{m-1}(pq) = 0$ (Benoumhani's Proposition)

$$r_m(pq) = pq\,r_{m-1}(pq) - r_{m-2}(pq) \qquad \text{(by recurrence)}$$
$$= -r_{m-2}(pq) < 0$$

Then the game terminates at the position $(u(m), v(m)) = (r_m(pq)a, r_m(pq)b)$.

One can similarly see that for $pq = 4\cos^2(\pi/m)$ and $\underline{\underline{m \text{ even}}}$, any game played from strongly dominant initial position (a, b) where we begin firing at node γ_2 will terminate at the same position after precisely m node firings as well. The argument when m is odd is entirely similar.

And when $pq \geq 4$, it is seen in a similar fashion that the game sequence $(\gamma_2, \gamma_1, \gamma_2, \ldots)$ does not terminate. //

- **Proposition** Consider the two-node game graph $\overset{\gamma_1}{\bullet} \xrightarrow{\;p\;} \xleftarrow{\;q\;} \overset{\gamma_2}{\bullet}$ with $p, q > 0$.

  Take a strongly dominant initial position (a, b), so $a, b > 0$.

  If $0 < pq < 4$ but $pq \neq 4\cos^2(\pi/m)$ for all integers $m \geq 3$, then there are convergent game sequences $(\gamma_1, \gamma_2, \gamma_1, \ldots)$ and $(\gamma_2, \gamma_1, \gamma_2, \ldots)$ of <u>different</u> lengths which can be played from initial position (a, b).

  Reference: See [P17]

- The upshot of the preceding Propositions / analysis of games on two-node game graphs is this: Eriksson's SC Theorem holds for all two-node game graphs.

- **Proposition** If Eriksson's SC Theorem holds for all two-node game graphs, it holds for all game graphs.

  <u>Proof:</u>

  1st, assume the "$\Rightarrow$" direction of Eriksson's SC Theorem for two-node game graphs. That is, assume that conditions 1, 2, and 3 are necessary for any two-node game graph $\overset{\alpha}{\bullet} \xrightarrow{\;p\;} \xleftarrow{\;q\;} \overset{\beta}{\bullet}$ to be an SC-graph. Now let $\mathcal{G} = (\Gamma, A)$ be any game graph, and assume that $\mathcal{G}$ is an SC-graph. We wish to show that for all $i \neq j$ in I_n, it is the case that conditions 1, 2, and 3 hold.

So for $i \neq j$ in I_n, choose an initial position λ with $\lambda_i, \lambda_j > 0$ and $\lambda_k << 0$ for all $k \neq i, j$. By our analysis of Cases 1, 2, 3 above, it must be the case that $-a_{ij} > 0$ and $-a_{ji} > 0$. If $a_{ij}a_{ji}$ is in the open interval $(0,4)$ and $a_{ij}a_{ji} \neq 4\cos^2(\pi/m)$ for all integers $m \geq 3$, then by the above Proposition there are game sequences from position λ with different lengths, violating strong convergence. So either $a_{ij}a_{ji} = 4\cos^2(\pi/m)$ for some integer $m \geq 3$ or else $a_{ij}a_{ji} \geq 4$.

2nd, assume the "$\Leftarrow$" direction of Eriksson's SC Theorem for two-node game graphs. Now let $\mathcal{G}$ be any game graph, and suppose $\mathcal{G}$ has an initial position for which there is a convergent game sequence leading to some terminal position μ and another legal play sequence of the same length that does not end at μ. In fact, let λ be an initial position for which there is a shortest pair of legal play sequences (s_1, s_2) each with length k so that s_1 is a game sequence terminating at some position μ and s_2 does not end at μ.

Say s_1 starts by firing a node γ_i. If s_1' is any other game sequence that also starts by firing node γ_i, then s_1' must have length k and terminate at μ. Otherwise, if we remove γ_i from the beginning of each of s_1 and s_1' and keep only $k' = \min(\text{length}(s_1)-1, \text{length}(s_1')-1)$ of the remaining nodes from each of s_1 and s_1', we get a pair $(\hat{s}_1, \hat{s}_1')$ that is shorter than (s_1, s_2).

Say s_2 starts by firing a node γ_j. Use reasoning similar to the previous paragraph to see that all legal play sequences from λ that start by firing node γ_j and have length k must not end at position μ.

So we conclude that γ_i and γ_j are distinct nodes of $\mathcal{G}$, and λ_i and λ_j are both positive. But now the fact that "any game sequence from λ that starts by firing node γ_i converges to μ" means that $0 \leq a_{ij}a_{ji} < 4$. Then $a_{ij}a_{ji} = 4\cos^2(\pi/m)$ for some integer $m \geq 2$. Then the sequences $(\gamma_i, \gamma_j, \gamma_i, \ldots)$ and $(\gamma_j, \gamma_i, \gamma_j, \ldots)$ — each of length m — can both be played legally from λ to some position λ'. Then we can find a shorter pair (s_1', s_2') from λ', with s_1' a game sequence of length less than k and terminating at μ and with s_2' a legal play sequence from λ' of the same length as s_1' but which doesn't end at μ. But this contradicts the fact the choice of (s_1, s_2) as a shortest pair.

So, we conclude that whenever $\mathcal{G}$ has an initial position λ for which there is a convergent game sequence of length k terminating at a position μ, then all game sequences from λ have length k and terminate at μ. That is, $\mathcal{G}$ is an SC-graph. //