# Admissible SC-Graphs

**Define**: $G = (\Gamma, A)$ is a connected SC-Graph.

**Definition**: $G$ is admissible if and only if there is a nontrivial dominant starting position $\lambda$ on $G$ such that there is a convergent game sequence from $\lambda$.

**Theorem**: A connected SC-Graph is admissible if and only if it is in one of the following mutually exclusive families of SC graphs:

$A_n (n \geq 1)$

$B_n (n \geq 2)$

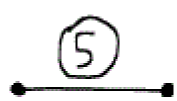

$D_n (n \geq 4)$

$E_n (n = 6, 7, 8)$

$F_4$


$G_2$


$H_2$




$H_3$

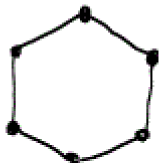
$H_4$ 

$I_2(m)$  $, m \geq 7$.

It is a "well-known" fact that $W = W(G)$ is finite if and only if $G$ is from the above list. (Chapter 2 of Humphrey's book),(Appendix C of Davis's book)

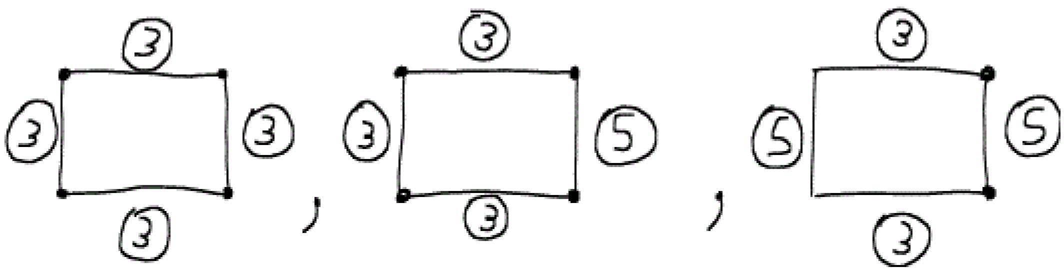
**Proof of Theorem**: We will use induction on the number of nodes to prove the "only if" part of this theorem. For $n=1$, $G = \bullet$, which is in the list. Now suppose this is also true for all positive integers $k \leq n$, for some positive integer $n$. Now let $G$ be an $n+1$ node connected admissible SC-Graph. We will break this up into two cases, "unital ON-cyclic" and "not unital ON-cyclic".

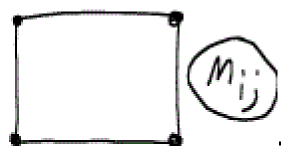
Case 1: Suppose $G$ is "unital ON-cyclic". Note that a cycle is an ON-cycle if all the $m_{ij}$'s are odd. Also note that a ON-cycle is unital if the product of the amplitudes in one direction around the cycle equals the product in the opposite direction. Lastly note that $G$ is unital ON-cyclic if every ON-cycle in $G$ is unital. Let $W = W(G)$. We know that $-U$ is the set of all positions from which there is a convergent game sequence by theorem[1], and where $U$ is the "Tits' Cone" which contains the dominant positions or "dominant chamber" denoted $D$. Since our $G$ is admissible, then we have a nontrivial dominant starting position $\lambda$ (so $\lambda \neq 0$ is in $D$) from which there is a convergent game sequence (so $\lambda$ is in $-U$). Therefore, $D \cap (-U) \neq \{0\}$, so $U \cap (-U) \neq \{0\}$. Then by the contrapositive of theorem[2], we have that $W(G)$ is finite. So by the well-known fact above, $G$ is from the list.

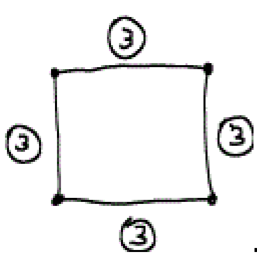
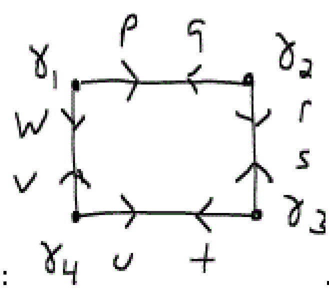
Case 2: Suppose $G$ is "not unital ON-cyclic". Therefore $G$ has an ON-cycle and hence $G$ has a least 3 nodes. Any cycle in $G$ must use all $n+1$ nodes of $G$. If not, then there is an admissible cyclic subgraph that has no more than $n$ nodes by theorem[3]. By the inductive hypothesis, this graph would have to be in the list, and there are no cycles in the list. For this same reason, $G$ must have the form of a "loop", like  ,with no other connecting edges. Therefore $G$ itself is just a simple cyclic graph of odd neighborly edges.

If $G$ has exactly 3 nodes, then from Dr. Donnelly's paper "Eriksson's number game on certain edge-weighted three-node cyclic graphs" and his proposition[4] , $G$ is not admissible. Thus $G$ must have $\geq 4$ nodes.

Suppose that $G$ has 4 nodes. Then the only three possibilities are:




Why? The other possibilities are similar to  . Here if $m_{ij}$ is odd and $m_{ij} \neq 3,5$ then the sub-graph  would not be on the list. We will work through these cases above.

Case  . We will use the following labeling diagram: 

. Note that $pq = rs = tu = wv = 1$. We will consider the fundamental starting position $\omega_1 = (1,0,0,0)$. According to Dr. Donnelly's paper , repeating $(\gamma_2 \circ \gamma_3 \circ \gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1) = F$ is a divergent game sequence. So let's check this.

$$\omega_1 = (1,0,0,0)$$

$$\gamma_1(\omega_1) = (-1, p, 0, w)$$

$$(\gamma_2 \circ \gamma_1)(\omega_1) = (-1+qp, -p, rp, w) = (0, -p, rp, w)$$

$$(\gamma_3 \circ \gamma_2 \circ \gamma_1)(\omega_1) = (0, rps-p, -rp, trp+w) = (0,0,-rp, trp+w)$$

$$(\gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1)(\omega_1) = (vtrp+vw, 0, utrp+uw-rp, -trp-w) = (vtrp+1, 0, uw, -trp-w)$$

$$(\gamma_3 \circ \gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1)(\omega_1) = (vtrp+1, suw, -uw, tuw-trp-w) = (vtrp+1, suw, -uw, -trp)$$

$$(\gamma_2 \circ \gamma_3 \circ \gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1)(\omega_1) = (vtrp+1+qsuw, -suw, rsuw-uw, -trp) = (1+vtrp+qsuw, -suw, 0, -trp)$$

Note that every firing in this sequence was "legal". We will now use the following substitutions: $vtrp = \prod_C$, $qsuw = \prod_C^{-1}$ to get,

$$(\gamma_2 \circ \gamma_3 \circ \gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1)(\omega_1) = (1+vtrp+qsuw, -suw, 0, -trp) = (1+\prod_C+\prod_C^{-1}, -p\prod_C^{-1}, 0, -w\prod_C).$$

We will fire this same sequence again to get:

$$(\gamma_1)(F)(\omega_1) = (-1-\prod_C-\prod_C^{-1}, p+p\prod_C+p\prod_C^{-1}-p\prod_C^{-1}, 0, w+w\prod_C+\mathrm{w}\prod_C^{-1}-w\prod_C) =$$

$$= (-1-\prod_C-\prod_C^{-1}, p+p\prod_C, 0, w+\mathrm{w}\prod_C^{-1})$$

$$(\gamma_2 \circ \gamma_1)(F)(\omega_1) = (-1-\prod_C-\prod_C^{-1}+qp+qp\prod_C, -p-p\prod_C, rp+rp\prod_C, w+\mathrm{w}\prod_C^{-1}) =$$

$$= (-\prod_C^{-1}, -p-p\prod_C, rp+rp\prod_C, w+w\prod_C^{-1})$$

$$(\gamma_3 \circ \gamma_2 \circ \gamma_1)(F)(\omega_1) = (-\prod_C^{-1}, srp+srp\prod_C-p-p\prod_C, -rp-rp\prod_C, trp+trp\prod_C+w+w\prod_C^{-1}) =$$

$$= (-\prod_C^{-1}, 0, -rp-rp\prod_C, trp+trp\prod_C+w+w\prod_C^{-1})$$

$$(\gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1)(F)(\omega_1) =$$

$$= (vtrp+vtrp\prod_C+vw+vw\prod_C^{-1}-\prod_C^{-1}, 0, utrp+utrp\prod_C+uw+uw\prod_C^{-1}-rp-rp\prod_C, -trp-trp\prod_C-w-w\prod_C^{-1}) =$$

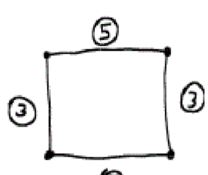
$$= (\prod_C + \prod_C^2 + vw + vw\prod_C^{-1} - \prod_C^{-1}, 0, uw + uw\prod_C^{-1}, -trp - trp\prod_C - w - w\prod_C^{-1})$$

$$(\gamma_3 \circ \gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1)(F)(\omega_1) =$$

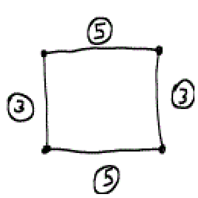
$$= (\prod_C + \prod_C^2 + vw + vw\prod_C^{-1} - \prod_C^{-1}, suw + suw\prod_C^{-1}, -uw - uw\prod_C^{-1}, tuw + tuw\prod_C^{-1} - trp - trp\prod_C - w - w\prod_C^{-1}) =$$
$$= (\prod_C + \prod_C^2 + 1, suw + suw\prod_C^{-1}, -uw - uw\prod_C^{-1}, -trp - trp\prod_C)$$

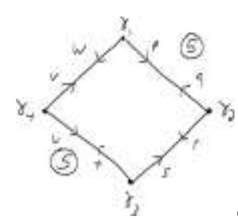
$$(\gamma_2 \circ \gamma_3 \circ \gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1)(F)(\omega_1) = (\prod_C + \prod_C^2 + 1, suw + suw\prod_C^{-1}, -uw - uw\prod_C^{-1}, -trp - trp\prod_C) =$$
$$= (psuw + psuw\prod_C^{-1} + \prod_C + \prod_C^2 + 1, -suw - suw\prod_C^{-1}, rsuw + rsuw\prod_C^{-1} - uw - uw\prod_C^{-1}, -trp - trp\prod_C) =$$
$$= (\prod_C^{-1} + \prod_C^{-2} + \prod_C + \prod_C^2 + 1, -p\prod_C^{-1} - \prod_C^{-2}, 0, -w\prod_C - w\prod_C^2) = (1 + \prod_C + \prod_C^{-1} + \prod_C^2 + \prod_C^{-2}, -p(\prod_C^{-1} + \prod_C^{-2}), 0, -w(\prod_C + \prod_C^2))$$

So we can see the pattern here. Note that all firing sequences have been legal and will continue to be legal if we continue this sequence, so this game will continue infinitely and therefore is divergent. We have shown this for one fundamental starting position. We know the other fundamental starting positions will be similar for this graph due to the symmetry of the graph. We also know it is sufficient to merely investigate the fundamental positions by lemma[5]. Thus this graph is inadmissible.

Case 

. From lemma[6] we know this graph is inadmissible.

Case 

. We will use the following labeling convention,  . Note:

$pq = ut = \dfrac{3+\sqrt{5}}{2}$, $rs = vw = 1$. We say a position $(a,b,c,d)$ meets condition (*) if $a > 0, b \geq 0, c \geq 0, d \leq 0, aw \oplus d \geq 0$, and $aprt \oplus brt \oplus ct \oplus d > 0$. Let the firing sequence $F = (\gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1)$. Call $\omega_1$ our initial position. Now we will check this firing sequence.

$$\omega_1 = (a, b, c, d)$$

$$(\gamma_1)(\omega_1) = (-a, pa \oplus b, c, wa \oplus d)$$

$$(\gamma_2 \circ \gamma_1)(\omega_1) = (qpa \oplus qb - a, -pa - b, rpa \oplus rb \oplus c, wa \oplus d)$$

$$(\gamma_3 \circ \gamma_2 \circ \gamma_1)(\omega_1) = (qpa \oplus qb - a, sc, -rpa - rb - c, trpa \oplus trb \oplus tc \oplus wa \oplus d)$$

$$(\gamma_4 \circ \gamma_3 \circ \gamma_2 \circ \gamma_1)(\omega_1) = (vtrpa \oplus vtrb \oplus vtc \oplus vd \oplus qpa \oplus qb, sc,$$

$$, utrpa \oplus utrb \oplus utc \oplus uwa \oplus ud - rpa - rb - c, -trpa - trb - tc - wa - d).$$

Notice that $A = \left( \frac{3+\sqrt{5}}{2} a \oplus bq \oplus v(aprt \oplus brt \oplus ct \oplus d) \right) > 0,$

$B = sc \geq 0,$

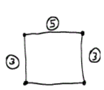
$C = u(atrp \oplus brt \oplus ct \oplus d) \oplus uaw - rpa - rb - c \geq 0,$

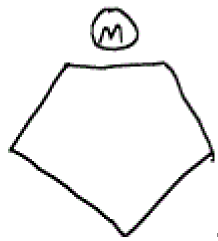
$Aw \oplus D = \left( \frac{3+\sqrt{5}}{2} - 1 \right) aw \oplus bqw > 0,$

$$Aprt \oplus Brt \oplus Ct \oplus D = \frac{3+\sqrt{5}}{2} aprt \oplus prtbq \oplus prtv(aprt \oplus brt \oplus ct \oplus d) \oplus tc \oplus$$

$$\oplus ut(atrp \oplus brt \oplus ct \oplus d) \oplus tuaw - trpa - trb - tc - aw - aprt - brt - ct - d > 0.$$

So $(A, B, C, D)$ meets condition (*).

The fundamental position $\omega_1 = (1, 0, 0, 0)$ meets condition (*), so it follows that we can apply firing sequence $F^\infty$ legally to $\omega_1$. If we fire $(\gamma_4 \circ \gamma_3 \circ \gamma_2)$ to the position $\omega_2 = (0, 1, 0, 0)$ we would get $(vtr \oplus q, 0, \frac{1+\sqrt{5}}{2} r, -tr)$. Since this position meets condition (*), then we can legally play $(F^\infty \circ \gamma_4 \circ \gamma_3 \circ \gamma_2)$ from $\omega_2$. Similarly $(F^\infty \circ \gamma_4 \circ \gamma_3)$ can be legally played from $\omega_3$ and $(F^\infty \circ \gamma_4)$ can be legally played from $\omega_4$. Thus  is inadmissible.

So we now know that if $G$ is "not unital ON-cyclic" then it must have $\geq 5$ nodes since we have ruled anything smaller out. Note here that we must have that each $m_{ij}$ equals 3 or else we would not be on the list. For example, suppose we have 

. Where here $m > 3$ and odd. We can see that if we leave out a node, the resulting sub-graph is not on the list. Then by theorem[3], the graph is inadmissible. So we must have a 3 on each edge. But then By lemma[7] this graph is inadmissible. Therefore there are no admissible graphs with $\geq 5$ nodes.

We have shown that there are no Case 2 (“not unital ON-cyclic”) graphs which are admissible. All Case 1 (“unital ON-cyclic”) graphs come from the list. So this completes the induction step and thus we have that a connected SC-Graph is admissible only if it is in one of the mutually exclusive families of SC graphs on our list.

Now we must show that if a graph is on our list then it is admissible. For proof by contradiction, suppose there is a graph on the list for which there is a divergent game sequence. This game sequence will look as follows: $(\gamma_{i_1}, \gamma_{i_2}, \ldots\ldots\ldots\ldots)$. Then for each product:

$$w_1 = s_{i_1}$$

$$w_2 = s_{i_2} s_{i_1}$$

$$w_3 = s_{i_3} s_{i_2} s_{i_1}$$

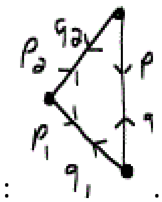
$$\vdots$$

By Eriksson’s reduce word theorem[8], we see that we would then have a infinite group because it would have elements that are arbitrarily long. This contradicts our “well-known” fact that $W = W(G)$ is finite if and only if $G$ is from our list. Thus if a graph is on our list of mutually exclusive families of SC graphs then every game sequence converges. **QED**

[1] "Eriksson's Tits Cone Convergence Theorem": $-U = \{\lambda \in V^* \mid$ there is a convergent game sequence from start position $\lambda\}$.

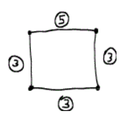
[2] Let $G$ be connected and unital ON-cyclic. If $W = W(G)$ is infinite, then $U \cap (-U) = \{0\}$.

[3] If a connected SC-Graph is admissible, then any connected SC-Subgraph is also admissible.

[4] Suppose $(\Gamma, A)$ is the following three-node SC-Graph: 

. Assume that all node pairs are odd-neighborly. Then $(\Gamma, A)$ is not admissible.

[5] An SC-graph is not admissible if for each fundamental position there is a
divergent game sequence.

[6] An SC-graph in the family 

is not admissible.

[7] Suppose that the underlying graph $\Gamma$ of an SC-Graph $(\Gamma, A)$ is a loop and that for any edge in $(\Gamma, A)$ the amplitude product is unity. Then $(\Gamma, A)$ is not admissible.

[8] Suppose $(\gamma_{i_1}, \ldots\ldots, \gamma_{i_p})$ is a legal firing sequence from some start position $\lambda$ on the SC-Graph $G = (\Gamma, A)$. Then $s_{i_p} \cdots\cdots s_{i_1}$ is a reduced expression in $W = W(G)$.